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THE ALGEBRA

OF

COPLANAR VECTORS AND TRIGONOMETRY

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OF

COPLANAR VECTORS AND TRIGONOMETRY

BY

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"Tant que l'algèbre et la géométrie ont été séparées, leurs progrès ont été lents et leurs usages bornés: mais lorsque ces deux sciences se sont réunies, elles se sont prêtées des forces mutuelles, et ont marché ensemble d'un pas rapide vers la perfection."—LAGRANGE.

"Die Arithmetik der complexen Zahlen ist der anschaulichsten Versinnlichung fähig . . . eine die Einsicht lebendiger machende und deshalb sehr zu empfehlende Versinnlichung."—GAUSS.

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TO THE MEMORY OF

AUGUSTUS DE MORGAN

FORMERLY PROFESSOR OF MATHEMATICS IN UNIVERSITY
COLLEGE, LONDON,

TO THE SPIRIT OF WHOSE TEACHING AND WRITINGS

THE AUTHOR, ONCE HIS PUPIL,

WOULD ASCRIBE

WHATEVER OF MERIT IS TO BE FOUND IN THE
FOLLOWING PAGES

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PREFACE

The present work is an attempt to supply the place of the *Trigonometry and Double Algebra* of De Morgan, which (published in 1849) has long been out of print and is now difficult to obtain.

It is in no sense however a reproduction of that work. De Morgan in the first part of his book treats of Trigonometry on the usual lines of the geometrical definitions, and in the second part shows the full significance of the results in the light of his *Double Algebra* or what is now often referred to as the Argand* interpretation. In the present treatise starting from the conception of a *Vector*, that is, a magnitude involving the two elements *Length* or *Quantity* and *Direction*, it is shown that, limiting the directions considered to one plane, we arrive at an Algebra identical, in all respects as to its laws, with the ordinary Algebra developed from the notion of number or magnitude involving the single element *Quantity;* but with this difference, that the former involves no unexplained symbols or, as they are commonly termed, "impossible" or "imaginary" quantities. Trigonometry follows naturally as the most important corollary or application of this Algebra, and this sequence I have adopted and endeavoured to express by the title "The Algebra of Coplanar

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* De Morgan mentions Argand's *Essai* (1806) as one of several works bearing on the subject, "which I have either not seen or cannot immediately obtain." See *Historical Note*, p. xx.

Vectors and Trigonometry." De Morgan says in his Preface, "The term *Double Algebra* has not yet obtained currency," and I think the same may still be said. I have therefore ventured to use the longer term, "the Algebra of Coplanar Vectors," as expressing more explicitly the extent and scope of the Algebra here developed. The complete treatment of vectors, not restricted to lie in one plane, has been shown in the Quaternions of Sir W. R. Hamilton, the Triple Algebra of De Morgan, the Ausdehnungslehre of Grassmann, and some other developments of Algebra, to require an Algebra differing in some of its laws from those of ordinary Algebra.

As the treatment of the subject in this work is somewhat novel, at any rate in the order of its development, a short summary of the successive chapters, calling attention to the salient points in each, will probably be found useful to the reader.

The Introduction contains a short summary of the principles and laws of ordinary Algebra, as founded on the notion of Number together with that of *opposite senses*, in which therefore the literal symbols denote *scalar*, that is, positive or negative numerical quantities. It is shown that number may here be regarded not simply as the *discrete* number pertaining to quantities *commensurable* with a supposed unit, but as number in the higher sense of Ratio (or Continuous Number as it may be termed) as defined by Euclid, in which the distinction of commensurable and incommensurable ceases to be relevant.

The first chapter explains the notions of a Vector and Vector-Aggregation (including in that term both Addition and Subtraction), and shows that these lead to the same simple laws as hold for the aggregation of numbers. The results are applied to the proof of a number of well-known Geometrical Theorems. As Vector-Aggregation does not require the Vectors to be limited to one plane, these applications include propositions in

Solid as well as in Plane Geometry. In an appendix it is shown that Velocities, Accelerations, Forces, &c. are Vector-Magnitudes to which the laws of Vector-Aggregation are applicable.

Ch. ii. treats of the *multiplication* of Vectors in *the same plane* and its consequences. The general multiplication of Vectors in space of three dimensions is beyond the scope of this treatise.

A definite vector in the supposed plane being assumed as the "prime vector," every other vector in the plane may be derived from it by extending (or contracting) its length in a given ratio (say $a : 1$) and turning it through a certain angle (say α) in one of the two opposite senses of rotation in the plane, and so may be denoted as (a, α). If from another vector (b, β), a third vector is derived by the same process, this last is $(ab, \alpha+\beta)$, and it is shown that the operation may be properly termed *multiplication*, so that we may write

$$(a, \alpha) \times (b, \beta) = (ab, \alpha + \beta):$$

and that the commutative, associative and distributive laws of ordinary Algebra are satisfied by the operation thus defined. The consequences of this definition are then developed, and it is shown that (using* the right angle as the angular unit) and denoting by i, the vector $(1, 1)$, or the unit vector whose inclination to the prime vector in the positive sense is a right angle, the vector (a, α) may be expressed as ai^{α}, or the *product* of two elements, a, termed its *tensor*, and i^{α}, its *versor*. The same vector may also be expressed in the form of a *complex number*, as $m + ni$, the component terms of which I call the *prōject* and *traject* respectively. Thus a vector may be regarded either as the *product* of *tensor* and *versor*, or as the *sum* of *project* and

---

* The right angle is at once the most natural, and the simplest and most convenient, unit in the development of the subject, and is here denoted by the symbol $\llcorner$. "Circular Measure" is only introduced later at a point where it naturally presents itself as having special advantages, but even then I have frequently used rectangular measure as in many cases presenting formulæ in a simpler form than that arising out of circular measure.

traject, and the relation of these two modes of presentation of a vector is the key to a great part of the farther development of the subject. The tensor is equal to the modulus of the complex number, $(a = \sqrt{m^2+n^2})$, and the versor to the quotient of the complex number divided by its modulus. The chapter concludes with some geometrical illustrations and applications.

Ch. iii. treats of the Trigonometrical or Circular Functions.

The *project* and *traject* of a *unit* vector (i^u) are defined as the *cosine* and *sine* respectively of its inclination (u) to the prime vector which leads to the fundamental equation

$$i^u = \cos u + i \sin u.$$

Hence we obtain at once the scalar relation $\cos^2 u + \sin^2 u = 1$, and the expressions for $\cos u$ and $\sin u$ in terms of versors; thus $2 \cos u = i^u + i^{-u}$ and $2i \sin u = i^u - i^{-u}$. Then defining the other circular functions with reference to sine and cosine, the usual relations of the functions of a single angle are proved in all their generality by direct algebraical processes.

Ch. iv. begins with De Moivre's Theorem, as a direct consequence of the fundamental equation $i^u = \cos u + i \sin u$. The ordinary formulæ of Trigonometry for two or more angles are then deduced; and the expansions of $\cos nu$, $\sin nu$ in powers of $\cos u$ or $\sin u$ the expressions for $\cos^n u$, $\sin^n u$ in terms of sines or cosines of multiples of u, and the fundamental relations of the sides and angles of a triangle established. The application of these last results to the Solution of Triangles, the Geometry of the Triangle, &c., is not pursued: for these the student is referred to other treatises on Trigonometry.

Ch. v. treats of Vector (or Complex) Indices and Logarithms.

Assuming that A^B, where A, B are both vectors or complex numbers, represents some vector, we consider first the case where $A = k$, a positive scalar, and $B = i$. Supposing then $k^i = ri^u$, the problem is to determine r and u. Assuming that there is a value of k, which makes $r=1$ and $u=1$, and denoting it by η, so that $\eta^i=i$, we find from the laws of Indices that

$$k^i=\eta^{i\ \log_\eta k}=i^{\ \log_\eta k}=\cos\ (\log_\eta k)+i\ \sin\ (\log_\eta k)$$

in which it remains to determine the value of η. After a discussion of the limits of $\sin z/z$ (which leads incidentally to the explanation of circular measure) and of $(a^z-1)/z$, when z vanishes, it turns out that $\eta=e^{\frac{\pi}{2}}$ and θ denoting the circular measure of $u^{\llcorner}$, the fundamental identity becomes

$$\cos u+i\sin u=i^u=\eta^{iu}=e^{i\theta},$$

whence the usual exponential expressions for the circular functions. The interpretation of A^B or $(ri^u)^{m+ni}$ easily follows: it turns out to be a function having an infinite number of values in vectorial geometrical progression, which are conveniently represented in a diagram.

The general Theory of Logarithms naturally follows, but is here discussed independently, and illustrated by diagrams and geometrical constructions.

Ch. vi. treats of Excircular (or Hyperbolic) Trigonometry.

The functions, which are usually called Hyperbolic, I have ventured to term Excircular, because they are in a certain sense, as we see in their geometrical interpretation, the Circular Functions *turned inside out.* They are not Hyperbolic, as opposed to Elliptic, but *Rectangular-Hyperbolic.* I hope to carry Mathematicians with me in giving the names of *Excircle* to the *Rectangular Hyperbola,* and *Excircular Functions* to the Functions treated of in this chapter. The functions are defined as those obtained by putting iu for u in the fundamental identities, and the formulæ corresponding to those established for Circular Functions are deduced. Their geometrical relation to the Excircle is then exhibited, and a number of the properties of the Excircle deduced. The chapter concludes with graphic constructions representing the vectors sin $(u+vi)$, cos $(u+vi)$, &c.

In Ch. vii. the fundamental properties of the roots of unity are discussed both as an instructive exercise in the application of the principles established in the previous chapters and for the sake of the importance of the analytical results.

Ch. viii. deals with the general properties of Infinite Series, and the conditions of their Convergency and Divergency, so far as seems necessary for an Elementary Treatise. It commences with a complete discussion of the Geometric Vector Series

$$1 + ri^u + r^2i^{2u} + r^3i^{3u} + \dots,$$

showing that it is convergent when r is less than 1 and divergent when r is greater than 1; but that when $r = 1$, although the sum of any number of terms is finite, it does not approach to a definite limit, but *fluctuates* (within a certain circle whose radius $= \frac{1}{2}$ cosec $\frac{u}{2}$), except only in the limiting case when $u = 0$ or the series is scalar, in which case it is divergent. All this is made evident by geometrical representation, which also furnishes a simple and interesting construction for the centre of convergence in the convergent case, and the centre and radius of the circle of fluctuation in the case of the fluctuating series.

An elementary discussion of the general properties of infinite series follows, aided by an illlustration which, I believe, will throw much light for the elementary student on some of the difficulties of the subject. The terms of a series are considered as the links of a chain, the scalar or tensor factor of each term determining the length of the link and the versor factor its inclination; then the sum of the series is the vector drawn from one end of the chain to the other, and convergency or divergency depends on whether this vector is finite or infinite, fluctuation corresponding to the case where the vector though not infinite is indefinite in length or direction. The discussion includes some remarks on the Continuity or Discontinuity of Infinite Series, and the conditions subject to which the fundamental laws of Algebra are applicable to them. The discussion is then confined to the *Power Series*, and tests for its Convergency, Divergency or Fluctuation established and applied to a number of examples.

Ch. ix. treats fully of the Binomial Theorem in its most general form, the Exponential and Logarithmic Series, and the Trigonometrical Series directly derived from each. Then follows the summation of various series by the help of these known series, and also of certain series by the method of differences, and the chapter concludes with an account of Bernouilli's numbers and certain series involving them.

Ch. x. treats of Factor Series, first as to their general properties with a Geometrical Illustration, and then deducing the factor series for sin u and cos u from the resolution of $z^n - a^n$ into factors. An extended form of Cotes's and De Moivre's Properties of the Circle is obtained, and applied to the geometrical interpretation of these series. The chapter concludes with series involving the sums of the inverse integral powers of the natural numbers, and by means of these Bernouilli's numbers are expressed, and the conditions of Convergency or Divergency of the series at the end of the previous chapter determined.

Ch. xi. deals in like manner with the expressions of cotan, tan, &c., in series of partial fractions.

Ch. xii. concludes the work with such an account of some fundamental properties of Rational and Integral Functions of a Complex Variable as naturally fall within the limits of an elementary Treatise. If z denote a vector in one plane, and w a vector in another plane, and $w = f(z)$, where fz is a rational and integral function of z of the n^{th} degree, (z, w being each drawn from a certain fixed origin in its plane,) for every point on the z plane there is one and only one corresponding point on the w plane, and it is shown conversely that for every point on the w plane there is *a* point, and if one, then n points, corresponding to it on the z plane. The n points corresponding to the origin or $w=0$, for which $fz=0$, are termed the *radical points* of the function. Cauchy's test for the number of radical points within given limits on the z plane is then established, in a simpler manner (I venture to believe) than that usually given. A short account of Conjugate Functions, as exemplified by Rational and Integral Functions, is then given with their fundamental Geometrical Relations. The chapter concludes with a detailed discussion of a Cubic Function with scalar coefficients, the different cases into which it resolves itself being illustrated by a number of diagrams.

In addition to specific acknowledgments in the text of the sources, whence I have derived matter which demanded such acknowledgment as hardly yet included within the limits of the recognised common property of all Mathematicians, I wish to acknowledge generally deep obligation to Prof. Chrystal's great work on Algebra, especially in the chapters relating to Infinite Series, for the fuller development of which beyond what falls properly within the limits of my own work Prof. Chrystal's treatise will form the best and most available guide. From Mr. Hobson's excellent Treatise on Plane Trigonometry also I have derived one or two valuable suggestions.

Messrs. Levitt and Davison's recently published *Elements of Plane Trigonometry*, in which the subject is very ably and accurately treated, has in its later chapters much in common with the present treatise, as was to be expected since like my own it is largely based on De Morgan's work. Their standpoint and the order in which they have developed the subject however differ essentially from those of the present work. It is perhaps well to note that in one point of notation in which we have independently hit on the same device, we have unfortunately carried it out in opposite directions. I have distinguished the *prime* value of an inverse function by using a capital initial letter (*e.g.* $\text{Tan}^{-1}x$), while the same written with a small initial denotes the general value ($\tan^{-1}x$): Messrs. Levitt and Davison indicate the prime value by the small, and the general value by the capital letter.

The present work assumes on the part of the student a competent knowledge of Elementary Algebra and Elementary Plane Geometry as usually understood, but not necessarily any previous knowledge of Trigonometry. It will probably, however, be most instructive to those who have already studied some elementary work in which that subject is treated on the usual geometrical basis.

R. B. H.

HARROW,
September 1892.

HISTORICAL NOTE

In the year 1806 there was printed at Paris an "opuscule" entitled *Essai sur une Manière de représenter les Quantités Imaginaires dans les constructions Géométriques*, without the name of the author, but to be obtained "chez Madame Veuve Blanc, Horloger, rue S. Honoré." The work appears to have remained unknown, at any rate unnoticed, by Mathematicians, until there appeared in the *Annales de Gergonne*, tome iv. 1813–1814, a note by J. F. Français on the same subject, which elicited from M. Argand* a letter addressed to M. Gergonne, accompanied by a copy of the *Essai*, with this alteration in manuscript on the title-page, "Chez M. Argand, rue de Gentilly, No. 12." The letter, which was published in T. iv. of the *Annales*, contains a summary of Argand's *Essai*, and was followed in T. v. by an article further developing his ideas.

A second edition of the *Essai* was published in 1874 with a preface by M. J. Hoüel (whence I have obtained the above information) and an appendix containing, besides Argand's papers in the *Annales*, notes of Français, Servois, and Gergonne, who took part in the discussion.

Though M. l'Abbé Buée, in a paper read June 20, 1805 and published in the *Phil. Trans.* for 1806, argued that the symbol $\sqrt{-1}$ should be regarded as expressing perpendicularity, he was not so successful in working out the consequences of this interpretation as Argand, for whom Hankel (quoted by Hoüel) justly claims the honour of being "the true founder of the theory of complex quantities in one plane," at any rate so far as priority and completeness are concerned.

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* Probably "Jean-Robert Argand, fils de Jacques Argand et de Eve Cauve," born at Geneva, 22 July, 1768. Little more is known of his personal history than that he was residing at Paris in 1813.

Argand concludes his *Essai* with the following remarkable passage :—

"La Méthode dont on vient d'exposer l'essai repose sur deux principes de construction, l'un pour la multiplication, l'autre pour l'addition des lignes dirigées : et il a été observé que ces principes *résultant d'inductions qui ne possèdent pas un degré suffisant d'évidence*, ils ne pouvaient, quant à présent, être admis que comme des hypothèses, que leur conséquences ou des raisonnements plus rigoureux pourront faire admettre ou rejeter."

It was reserved for a series of writers during the first half of the present century (several of whom appear to have arrived independently at the same interpretations) to supply, in connection with discussions on the foundations of Algebra, the true logical basis which Argand felt to be wanting for his interpretations. The following list of the principal writings of this class is taken from De Morgan's *Trigonometry and Double Algebra.*

List of some writings on the subject of Algebra, *in which the peculiar Symbols of Algebra are discussed.*

London, 1685, folio. John Wallis. *A Treatise of Algebra, both historical and practical.* Reprinted in Latin, with additions, in the second volume of *Wallis's Works*, Lond. : 1693, folio.

Naples, 1687, folio. Giles Francis de Gottignies. *Logistica Universalis.*

London, 1758, 4to. Francis Maseres. *A Dissertation on the use of the Negative Sign in Algebra.*

London, 1796, 8vo. William Frend. *The Principles of Algebra.**

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* An opponent not only of imaginary but of negative quantities. Perhaps this work suggested M. Buée's memoir. I have a letter in my possession

Cambridge, 1803, 4to. Robert Woodhouse. *The Principles of Analytical Calculation.*

Philosophical Transactions for 1806. M. l'Abbé Buée. *Mémoire sur les quantités Imaginaires.* (Read June 20, 1805.) See also the review of this in vol. xii. of the *Edinburgh Review,* April—July, 1808 (written by Playfair).

London, 1817, 4to. Benjamin Gompertz. *The Principles and Application of Imaginary Quantities,* Book I., *to which are added some observations on porisms……*

London, 1818, 4to. Benjamin Gompertz. *The Principles and Application of Imaginary Quantities,* Book II., *derived from a particular case of functional projections……*

Paris, 1828, 8vo (small). C. V. Mourey. *La vraie théorie des Quantités Négatives, et des Quantités Prétendues Imaginaires. Dédié aux amis de l'évidence.*

Cambridge, 1828, 8vo. John Warren. *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities.*

Philosophical Transactions for 1829. John Thomas Graves. 'An attempt to rectify the inaccuracy of some logarithmic formulæ.' (Read December 18, 1828.)

Philosophical Transactions for 1829. John Warren. Consideration of the objections raised against the geometrical representation of the square roots of negative quantities. (Read Feburary 19, 1829.) The same volume contains John Warren 'On the geometrical representation of the powers of quantities, whose indices involve the square roots of negative quantities.' (Read June 4, 1829.)

Cambridge, 1830, 8vo. George Peacock. *A Treatise on Algebra.*

from M. Buée to Mr. Frend, dated June 21, 1801, by which it appears that the former was desired by a gentleman in whose house he was living (as tutor, perhaps) to write a private reply to Mr Frend's objections. This letter evidently contains the germs of the views which he afterwards published. See the Annual Report of the Royal Astronomical Society for 1842. According to Dr. Peacock, M. Buée is the first formal maintainer of the geometrical signification of $\sqrt{-1}$.

Cambridge, 1837, 8vo. Anonymous [Osborne Reynolds]. *Strictures on certain parts of 'Peacock's Algebra,'* by a Graduate.

Philosophical Transactions for 1831. Davies Gilbert. 'On the nature of negative and of imaginary quantities.' (Read November 18, 1830.)

London, 1834, 8vo. *Report of the Third Meeting of the British Association for the Advancement of Science.* This volume contains George Peacock, 'Report on certain branches of analysis,' a most valuable historical discussion on, among other things, the advance of Algebra. I cite from it the following works, which I have either not seen, or cannot immediately obtain.

Paris, 1806, Argand. *Essai sur la manière de représenter les Quantités Imaginaires dans les constructions géométriques.* Also papers or observations by François, Argand, Servois, Gergonne, in the *Annales des Mathématiques for* 1813 (and I suppose the following year). Also a paper on the arithmetic of impossible quantities, by Playfair, in the *Philosophical Transactions for* 1778; with a Reply, by Woodhouse, in the same work for 1802, entitled 'On the necessary truth of certain conclusions obtained by aid of imaginary expressions.'

London, 1836, 8vo. Anonymous [George Peacock]. *A Syllabus of a Course of Lectures upon Trigonometry and the Application of Algebra to Geometry.*

London, 1837, 8vo. A. De Morgan. Elements of Algebra. 2nd edition.

London, 1837, 8vo. A. De Morgan. *Elements of Trigonometry and Trigonometrical Analysis, preliminary to the Differential Calculus......*

Edinburgh Philosophical Transactions, Vol. XIV., Part 1. D[uncan] F[orbes] Gregory. 'On the real nature of Symbolical Algebra.' (Read May 7, 1838.)

Ladies' Diary. London, 1839, 8vo (small). Thomas White. 'On the algebraical expansion of quantity,......and on the symbol

$\sqrt{-1}$, which is usually considered to denote impossible or imaginary quantity' (at page 59).

Cambridge Philosophical Transactions, Vol. VII., Part 2. A. De Morgan. 'On the Foundation of Algebra. (Read Dec. 9, 1839.)

Cambridge Philosophical Transactions, Vol. VII., Part 3. A. De Morgan. 'On the Foundation of Algebra,' No. II. (Read Nov. 29, 1841.)

Paris, 1841, 8vo. M. F. Vallès. *Etudes Philosophiques sur la science du calcul.* Première Partie. No more yet published.

Cambridge Philosophical Transactions, Vol. VIII., Part 2. A. De Morgan. 'On the Foundation of Algebra,' No. III. (Read Nov. 27, 1843.)

Cambridge, 1842 and 1845, 8vo. George Peacock. *A Treatise on Algebra.* Vol. I. Arithmetical Algebra. Vol. II. Symbolical Algebra and its applications to the geometry of position.

London, 1843, 12mo. Martin Ohm [translated by Alexander John Ellis]. *The Spirit of Mathematical Analysis, and its relation to a logical system.*

To the foregoing list should be added the great name of Gauss, who in a notice ("Anzeige"), dated 1831, April 23, of his "Theoria Residuorum Biquadraticorum. Commentatio secunda" (*Werke,* ii. pp. 174–178) speaks of the Arithmetic of Complex Numbers as "der anschaulichsten Versinnlichung fähig," and concisely, but very clearly, lays down the principles of such "Versinnlichung" by geometrical representation. He observes that, although in his memoir he has adopted a purely arithmetical treatment, he has also given sufficient indications, for the reader who thinks for himself ("selbstdenkende Leser"), of "eine die Einsicht lebendiger machende und deshalb sehr zu empfehlende Versinnlichung." There can be little doubt that Gauss made great use of the geometrical representation of "imaginaries" ("eine wenig schickliche Benennung") in his researches, although he presented his results for the most part in what he terms a purely arithmetical form.

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THE ALGEBRA

OF

COPLANAR VECTORS AND TRIGONOMETRY

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INTRODUCTION

1. *Algebra* is a science which originated historically in the notion of *number*, and the expression by words (for which symbols were gradually introduced), of the simple fundamental operations on numbers; namely, the direct operations of *addition* and *multiplication* with their respective inverses, *subtraction* and *division*.

The lectures on Algebra, which Newton delivered as Lucasian Professor at Cambridge, were published under the title of *Arithmetica Universalis*, and it is from this point of view of "Generalized Arithmetic" that the subject is still treated, and rightly so, in elementary works on algebra.

The modern developments of algebra, embracing all organized systems of symbols combined according to definite laws, extend to subject-matter of which number or quantity constitutes but one element, or in which this element is entirely absent, and the letters and symbols denote *operations* or *forms* only. It is with one of these developments, in which the laws and resulting formulæ are those of ordinary algebra, but the meanings of the symbols are extended by the introduction of a new element, so that no uninterpretable forms or, so-called "impossible" or "imaginary" quantities remain, that this treatise is concerned.

The method it is proposed to adopt is, to start with the notion of a *Vector;* from this to deduce the laws of combination of vectors in the same plane, showing that these laws are identical in their symbolical expression with those of pure number; and then to show, among other results, that Trigonometry emerges as an immediate consequence and thus becomes affiliated to, or rather included in, this branch of algebra.

2. In order to prepare the way for our special subject, a short review of the foundations and scope of ordinary algebra, with its consequent limitations, may fitly here be introduced.

Algebra starts with the use of letters (a, b, x, y, &c.) to denote numbers unspecified, and the signs of the four fundamental arithmetical operations. Thence are derived the following fundamental laws:

I. *The Associative Law:*

α. of Terms.

That in an aggregate of terms any pair or set of consecutive terms may be grouped into a single term:

e.g. $$a+b-c=(a+b)-c=a+(b-c).$$

β. of Factors.

That in a product of factors any pair or set of consecutive factors may be grouped into a single factor:

e.g. $a \times b \times c = ab \times c = a \times bc.$ $a \times b \div c = (a \times b) \div c = a \times (b \div c).$

II. *The Commutative Law:*

α. of Terms.

That the sum or aggregate of a number of terms (whether positive or negative) is independent of the order of aggregation:

e.g. $$a+b-c=a-c+b=-c+b+a, \text{ \&c.}$$

β. of Factors.

That the product of factors is independent of the order of multiplication (including under this term *division*, as *multiplication* by the *reciprocal*),

e.g. $$ab = ba, \quad a \cdot \frac{1}{b} = \frac{a}{b} = \frac{1}{b} \cdot a.$$

III. *The Distributive Law :*

That the product (or quotient) of an aggregate of terms multiplied (or divided) by any factor is the aggregate of the products (or quotients) of each term multiplied (or divided) by the same factor :

e.g. $$m(a \pm b) = ma \pm mb, \quad \frac{a \pm b}{m} = \frac{a}{m} \pm \frac{b}{m}.$$

IV. *The Law of Signs :*

α. for the signs + and −,

$$+(+a) = +a, \quad +(-a) = -a, \quad -(+a) = -a, \quad -(-a) = +a.$$

β. For the signs × and ÷,

$$\times(\times a) = \times a, \quad \times(\div a) = \div a, \quad \div(\times a) = \div a, \quad \div(\div a) = \times a.$$

V. *The Laws of Indices :*

$$1.\ a^m \cdot a^n = a^{m+n}. \quad 2.\ (a^m)^n = a^{mn}. \quad 3.\ (ab)^m = a^m b^m,$$

which are rendered complete and general by the interpretations

$$\alpha.\ a^{\frac{m}{n}} = \sqrt[n]{a^m}, \quad \beta.\ a^{-n} = \frac{1}{a^n}, \quad \gamma.\ a^0 = 1.$$

3. These laws are the necessary and sufficient basis of ordinary algebra as a formal science, so that from them alone all the results of algebra are deducible.

It is an essential condition of such a science that its laws should be *universally* true, without exceptions, and it is a consequence of this condition that, from whatever primary meanings for the symbols its laws may be derived, any previously undefined combination of symbols must be interpreted in accordance with these laws, and not arbitrarily. Hence results the principle, enunciated by Peacock and termed by him the "Principle of the Permanence of Equivalent Forms," which asserts that * "all results of algebra which are general in form, though they have been established as true only for restricted meanings of the symbols, must also be true when the symbols are general in value as well as in form."

4. For a detailed discussion of these laws as based on the notion of number the student is referred to the treatises of Chrystal, C. Smith, Aldis, or Hall and Knight, to which may be added the fresh and original treatment of Oliver and Waite.

It will be well however to summarize here the successive stages in a *logical sequence*,† by which the laws of algebra may be conceived to be derived from the notion of number.

5. First, regarding the signs $+$, $-$, $\times$, $\div$ as signs of *operation* only, we may consider the literal symbols as advancing through the following stages.

1°. Where the letters denote integers.

Here the laws, with such restrictions as the meanings of the letters impose, are very obviously deducible, and are in fact the laws on which all arithmetical operations are based. The expression $a-b$, where b is greater than a, is unintelligible or "impossible," as is indeed the form $\frac{a}{b}$ or $a \div b$, unless a is an exact multiple of b.

2°. Where the letters denote finite fractions, including integers as a particular case.

---

* This must be understood with some qualification, when the results involve *infinite* series. See C. viii.

† The *logical* order in any science is generally neither the *historical* order of development, nor the exact order in which the student will have acquired his knowledge. It is rather an order in which on a review of the subject as a whole after a certain familiarity with it has been obtained, the fundamental principles and the consequences deduced from them may be conveniently arranged so as to show their affiliation and interdependence.

The laws, as far as regards terms, are deducible exactly as in the previous case for integers, but those involving factors depend on a suitable extension of the meaning of multiplication, when the multiplier is a fraction. Such extension may be made by adopting the following general definition of multiplication.

Multiplication consists in doing with the multiplicand what must be done with the unit to form the multiplier.

From this it follows that to multiply by a fraction (*e.g.* $\frac{2}{3}$) is to form the same fraction ($\frac{2}{3}$) of the multiplicand. With this extension of meaning the laws involving factors also are readily proved. The laws of indices also assumed to be true for fractional indices lead to the interpretation of $a^{\frac{m}{n}}$ as necessarily denoting the nth root of a^m or $\sqrt[n]{a^m}$.

3°. Where the meaning of the letters is still further extended so as to include *incommensurable* quantities, and thus to justify their use as representing the quantity, relatively to a given unit, of any magnitude whatsoever.

Every incommensurable number lies between two finite fractions, as $\frac{m}{n}$ and $\frac{m+1}{n}$, whose difference $\frac{1}{n}$ can be made as small as we please, and on the assumption that what is true of two magnitudes which can be made as near to one another as we please is true of any magnitude which always lies between them, the same laws which have been established for finite fractions are seen to hold good for incommensurable magnitudes also.

This last extension from the *commensurable* to the *incommensurable* by treating the latter as attainable as a *limit* from the former is necessitated by the impossibility of completely representing *continuous magnitudes* by numbers.

Number is essentially *discrete* or *discontinuous*, proceeding from one value to the next by a finite increment or jump, and so cannot, except in the way of a limit, represent, relatively to a given unit, a continuous magnitude for which the passage from one value to another may always be conceived as a *growth* through every intermediate value.

In Euclid's Fifth Book we have the development of the general notion of *ratio* for magnitudes generally, whether continuous or discrete. Thus treated, ratio, while including the notion of number, is independent of the distinction of commensurable and incommensurable, and is applicable to the complete and exact measurement of any magnitude relatively to a given unit. It is instructive to see how the laws of algebra result from this higher conception of ratio, or as it might be termed, *continuous number*, and this we proceed summarily to indicate in the following articles, which, as an alternative mode of deduction of those laws, may be omitted without loss to the sequence of the argument.

6. α. Let A, B be two magnitudes of the same kind, and therefore having a ratio, and let the ratio $A : B$ be denoted by a : then if U denote a unit or standard quantity of some particular continuous magnitude (as *length* or *time*), a magnitude X of the same kind can always be found such that $X : U = A : B$, whence $a = X : U$.

β. Then the associative and commutative laws for terms being obviously true for the magnitudes X, Y, Z..., they are also true for the ratios a, b, c... which measure them with respect to a common unit U, if we define $a \pm b$ or $(X : U) \pm (Y : U)$ as equivalent to the single ratio $X \pm Y : U$.

γ. For factors, the product ab must be regarded as the ratio compounded of the ratios a and b.

Then if $a = X : U$, $b = Y : U$, $c = Z : U$,

$$Y' \text{ can always be found such that } Y' : Z :: Y : U,$$

$$\text{and } X' \text{ such that } X' : Y' :: X : U,$$

$$\text{and therefore } a = X' : Y',\ b = Y' : Z,\ c = Z : U.$$

Hence, the ratio compounded of $A : B$ and $B : C$ being, according to Euclid's definition, $A : C$, we have

$$ab = (X' : Y') \,.\, (Y' : Z) = X' : Z, \text{ and } bc = (Y' : Z) \,.\, (Z : U) = Y' : U,$$

whence

$$ab \,.\, c = (X' : Z) \,.\, (Z : U) = X' : U \text{ and } a \,.\, bc = (X' : Y') \,.\, (Y' : U) = X' : U,$$

so that

$$ab \,.\, c = a \,.\, bc$$

which proves the associative law for factors.

δ. Also since $Y' : Z :: Y : U$, alternately $Y' : Y :: Z : U$, hence

$$bc = (Y : U) \,.\, (Z : U) = (Y' : Z) \,.\, (Z : U) = Y' : U$$

and

$$cb = (Z : U) \,.\, (Y : U) = (Y' : Y)\,(Y : U) = Y' : U,$$

therefore

$$bc = cb,$$

which proves the commutative law.

ε. If a, a' denote respectively the ratio $X : U$ and its reciprocal ratio $U : X$, then $aa' = (X : U) \,.\, (U : X) = X : X = 1$ or $a' = \frac{1}{a} \cdot$

Also since a, b may be written as $a : 1$, $b : 1$ respectively, if $\frac{a}{b}$ denote the ratio $a : b$ and $bb' = 1$, $\frac{a}{b} = (a : 1) \,.\, (1 : b) = (a : 1) \,.\, (b' : 1) = a \,.\, \frac{1}{b}$ or ab'.

Hence the quotient a/b being the same as the product of a and the reciprocal of b, the laws proved above for multiplication hold also for the inverse, division.

ζ. To prove the distributive law.

Let $a = X : U$, $b = Y : U$, and $m = Z : U$, and let Z', Z'' be so taken that

$$Z' : X :: Z : U, \text{ and } Z'' : Y :: Z : U,$$

then also

$$Z' \pm Z'' : X \pm Y :: Z : U.$$

Hence

$$\begin{aligned} m\,(a \pm b) &= (Z : U) \,.\, (X \pm Y : U) \\ &= (Z' \pm Z'' : X \pm Y) \,.\, (X \pm Y : U) \\ &= Z' \pm Z'' : U \\ &= (Z' : U) \pm (Z'' : U) \\ &= (Z' : X) \,.\, (X : U) \pm (Z'' : Y) \,.\, (Y : U) \\ &= ma \pm mb. \end{aligned}$$

η. The same reasoning which establishes the laws of signs for numbers is applicable to magnitudes generally and their complete measures by ratio, and the same is true of the laws of indices, so long as the indices are commensurable numbers. The form a^m, however, where m is incommensurable, can hardly be interpreted, except as a limiting case deducible from that where m denotes a finite fraction.

7. The laws of algebra having thus been shown to result from the widest conception of number or ratio while the signs + and − denote merely the operations of addition and subtraction, a farther step in their generalization consists in showing that these signs may be regarded without inconsistency as *signs of affection* no less than as *signs of operation.*

There are certain classes of magnitudes which may be called *purely quantitative* magnitudes, of which we can conceive the indefinitely continued diminution by subtraction down to annihilation, but for which, the zero being attained, no farther progress of the same kind is possible. If a, b denote the quantities relatively to a given unit, of two amounts of a purely quantitative magnitude, $a-b$ is "impossible" when $b>a$, and $-a$ by itself is unmeaning.

The *population* of a country or the strength of an army is such a magnitude. It may be diminished to annihilation, but here the process of necessity stops. So the *mass* or *quantity of matter* in a given space may be indefinitely reduced, but a zero or vacuum forms a definite limit. We can "draw the well dry," but cannot continue the exhaustion. The same is true of *heat* or *energy.* The absolute zero of temperature is a definite ideal limit, beyond which abstraction of heat is impossible.

Thus *population, mass, heat* or *energy* are instances of purely quantitative magnitudes, and the first has the further limitation that it can be expressed only by integral numbers, so that a fractional answer to a question as to the population of a place is an "impossibility."

There are however other, and far more numerous, kinds of magnitudes, which are termed *scalar* magnitudes, for which the process of diminution need not be regarded as terminating with annihilation.

Thus a man may not only expend all his money, but by continuing the process may run into debt. A man may go to a certain distance north of a place, but as he diminishes that distance by returning south he need not stop where he started, but may proceed indefinitely southward from his starting point.

Instances such as these, which may be indefinitely multiplied, at once suggest an interpretation for $a-b$, where $b > a$, and thus lead to the general conception of a negative quantity as intelligible *per se*, and denoting the exact opposite of some given quantity, and of the suitability of the signs $+$ and $-$ to express this opposition of affection in the literal symbols to which they are prefixed.

It should be observed that this extended meaning of the signs $+$ and $-$ is not exclusive of, but may be used in conjunction with, their original meanings as signs of operation: for in fact subtraction may be regarded as reversing the affection and adding the resulting quantity.

So also multiplication by a negative factor becomes multiplication by the number with reversal of the affection, and a negative index is found by interpretation, subject to the condition of the laws of indices remaining true, to denote the reciprocal of the same factor with a positive index.

8. The fundamental laws are now completely intelligible without any restriction as to "impossible" cases, but the development of algebra on this basis leads to new combinations of symbols, which present fresh difficulties of interpretation. It is found that every algebraical expression is reducible to the form $a+b\sqrt{-1}$, where a and b are completely intelligible, or, as they are termed, "possible" or "real" quantities, but $\sqrt{-1}$ is unexplained and is a purely symbolical representation of something, neither a positive nor a negative quantity, whose square is equal to -1, and so is termed an "impossible" or "imaginary" quantity. No further extension of the fundamental notion of *number* or *ratio* is possible, and thus ordinary algebra remains burthened with the purely symbolical form $a+b\sqrt{-1}$, termed a *complex number*, of which no intelligible interpretation can be given.

9. It has been our object in the preceding discussion to bring clearly into view the following fundamental facts and principles of algebra as a symbolical calculus.

1°. That the results of algebra are not *directly* the consequences of the meanings assigned to the literal symbols, but only *indirectly* through the fundamental laws of combination deducible from those meanings.

2°. That for a given system * of algebra the fundamental laws of combination must be true without any exceptions, so that any extension of meaning of the literal symbols, or interpretation of previously unexplained combinations, must be made subject to the condition that these laws remain unchanged, save by the removal of some previous restriction on their generality.

3°. That the "Principle of the Permanence of Equivalent Forms" is a consequence of the universality of the symbolical laws: and that therefore it follows from the fundamental principles of algebra that (for instance) the Binomial Theorem having been proved for the case where the index is a positive integer, it requires no farther proof to show that it is also true for any index,† positive or negative, integral or fractional. Euler's proof is only an application of this principle to a special case, in which the consequence is, or is supposed to be, more obvious to the elementary student than in the general form above stated.

4°. That formulæ in which some or all of the letters denote complex numbers, must be regarded as true symbolically, but uninterpretable, or rather waiting for a suitable interpretation.

5°. That the "impossible" or "imaginary" quantities of algebra are only "impossible" or "imaginary" *relatively* to the meanings of the literal symbols which have been assumed, and that as by the extension of those meanings several "impossibilities" have disappeared, so by some further extension the remaining impossibility, the complex number, may cease to be uninterpretable.

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* There are systems of algebra, in which the fundamental laws are not identical with those of ordinary algebra. For instance, in the algebra of quaternions, the commutative law ($ab = ba$) is not true.

† Subject however to certain limitations, when the number of terms of the series is infinite, as shown in C. viii.

10. The magnitudes considered in ordinary algebra are either purely quantitative, that is, such as can be represented by number or ratio only; or scalar, that is, such as can be represented by number or ratio with the addition of the signs + and − to indicate two opposite senses, in either of which the magnitude may be conceived to be indefinitely extended.

The type and representative of all scalar magnitudes is length measured forwards or backwards along an indefinitely extended straight line. Ordinary algebra may thus be said to be *linear* or *one-dimensional*, and the appearance of the symbol $\sqrt{-1}$, which being neither a positive nor negative quantity is as a numerical or scalar magnitude uninterpretable, suggests the possibility that its interpretation may be found in *planar* or *two-dimensional* space. That this is the fact together with the consequences resulting therefrom we proceed to show in the chapters which follow.

CHAPTER I

VECTORS AND VECTOR-AGGREGATION

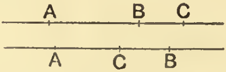
1. *Definition of vector.*

A *vector* is a magnitude which is completely defined by the two elements *length* or *distance*, and *direction.*

Thus a finite straight line drawn from one point to another is a vector.

2. *Vectors in the same line. Scalars.*

If A, B are two points, the straight line drawn from A to B, or the *step* which *carries* (*vehit*) a point from A to B along the straight line AB, is the vector AB, which we shall denote thus, $\overline{AB}$. The opposite step from B to A, since it exactly neutralizes the step from A to B, may be expressed as $-\overline{AB}$, so that $\overline{BA}=-\overline{AB}$, or $\overline{AB}+\overline{BA}=0$.



If C be a third point in the same straight line, it follows that $\overline{AB}+\overline{BC}=\overline{AC}$, or $\overline{AB}+\overline{BC}+\overline{CA}=0$. The student should verify this for any relative positions of A, B, C along the line.

Thus if vectors are considered which lie along a given line, they differ only in the elements of *quantity* and *sense*, and so are completely represented by the positive and negative quantities of ordinary algebra.

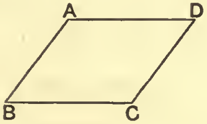
These positive and negative quantities, the "possible" or "real" quantities of ordinary algebra, are termed "*scalars.*"

3. *Equality of vectors.*

Lines which are parallel to a given straight line, and therefore to one another, are said to have the same *direction.* Vectors which in this sense have the same direction, may be measured one towards one end, and the other towards the other end of the line. For this opposition of direction along the same line we shall use the word *sense,* and the opposite *senses* will be indicated by the signs + and −, as with scalars.

Two vectors may then be said to be equal, when they have the same quantity (or length) and the same *sense* in the same *direction.*

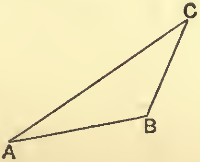
Thus if $ABCD$ is a parallelogram, $\overline{AB}$ and $\overline{DC}$ are equal vectors, or the step from A to B is equal to that from D to C, but $\overline{AB}$ and $\overline{CD}$, though equal in other respects, differ in sign, so that $\overline{AB} = -\overline{CD} = \overline{DC}$, and $\overline{AB} + \overline{CD} = 0$, or $\overline{AB} - \overline{DC} = 0$.



We shall see that, when we come to define direction by means of an angle, the *sense* of a vector may be merged in its *direction* by properly measuring the angle, but at present it will be convenient to retain the distinction.

4. *Aggregation of vectors.*

The fundamental law for the aggregation (under which term we include both addition and subtraction) of vectors is the following.



If A, B, C be any three points, the aggregate or sum of the vectors $\overline{AB}$ and $\overline{BC}$, or the steps from A to B and from B to C is the vector $\overline{AC}$ or the step from A to C, so that using the sign $+$ to denote this aggregation or addition,

$$\overline{AB}+\overline{BC}=\overline{AC}.$$

From this, since $\overline{BC}=-\overline{CB}$, $\overline{AB}-\overline{CB}=\overline{AC}$,

therefore $$\overline{AB}-\overline{CB}+\overline{CB}=\overline{AC}+\overline{CB},$$

or $$\overline{AB}=\overline{AC}+\overline{CB},$$

in accordance with the definition above.

Also $$\overline{AB}+\overline{BC}+\overline{CA}=\overline{AC}+\overline{CA}=0,$$

or the vector-sum of the sides of a triangle taken in order in making a circuit of the perimeter is zero.

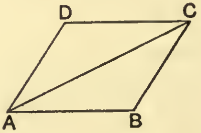
Hence we infer that the signs $+$ and $-$ connecting vectors may be used as in ordinary algebra, and to denote either signs of affection, or sense, or signs of operation.

5. *Associative and commutative laws of terms.*

Let AC be a diagonal of the parallelogram $ABCD$,

Then from the foregoing

$$\overline{AC}=\overline{AB}+\overline{BC}, \text{ and also } \overline{AC}=\overline{AD}+\overline{DC},$$



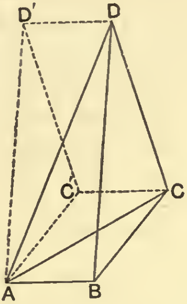
but by the definition of a vector, $\overline{AD}=\overline{BC}$, and $\overline{DC}=\overline{AB}$,

therefore $$\overline{AC}=\overline{BC}+\overline{AB},$$

so that $$\overline{AB}+\overline{BC}=\overline{BC}+\overline{AB}.$$

Also since $\overline{BC}=-\overline{CB}$, $\overline{AB}-\overline{CB}=-\overline{CB}+\overline{AB}$.

Hence it appears that in the aggregation of vectors the order of the terms is indifferent, or the commutative law is true.



Further, let $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ be any three vectors, and let $\overline{C'C} = \overline{D'D} = \overline{AB}$, $\overline{AC'} = \overline{BC}$, $\overline{C'D'} = \overline{CD}$ as in the figure: then

$$(\overline{AB} + \overline{BC}) + \overline{CD} = \overline{AC} + \overline{CD} = \overline{AD},$$

and also

$$\overline{AB} + (\overline{BC} + \overline{CD}) = \overline{AB} + \overline{BD} = \overline{AD}.$$

Again $\overline{AB} - \overline{CB} + \overline{CD} = (\overline{AB} - \overline{CB}) + \overline{CD} = \overline{AC} + \overline{CD} = \overline{AD}$,

and $\overline{AB} - \overline{CB} + \overline{CD} = \overline{AB} - (\overline{CB} - \overline{CD}) = \overline{AB} - \overline{DB} = \overline{AD}$.

Hence in the aggregation of vectors, any two terms may be grouped into a single term with due regard to the law of signs, or the associative law is true.

From these laws it follows, denoting the three vectors by α, β, γ respectively, that

$$\alpha + \beta + \gamma = \beta + \gamma + \alpha = \gamma + \alpha + \beta = \text{\&c.}$$

as may easily be verified from the figure, as thus:

$$\beta + \gamma + \alpha = \overline{AC'} + \overline{C'D'} + \overline{D'D} = \overline{AD} = \alpha + \beta + \gamma.$$

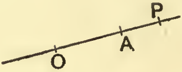
It may be left to the student to show that $\overline{BD'} = \beta - \alpha + \gamma$; and, completing the parallelepiped of which AB, BC, CD are edges, to express the vectors joining any two points of the figure in terms of α, β, γ.

6. *Application to geometry.*

From the simple laws of aggregation of vectors, a large body of geometrical truths may be readily deduced. With the view of familiarizing the student with the notion of vectors and their aggregation, the remainder of this chapter will be devoted to the investigation by this method of some well-known geometrical theorems. It will not be necessary here, as it will be found to be in the following chapters, to limit ourselves to vectors in one plane.

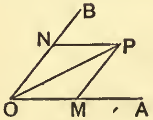
7. *Vector of any point expressed in terms of given vectors.*

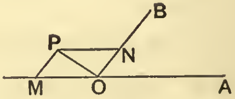
Let the vector $\overline{OA}$ be denoted by α, and let P be any point in the line OA indefinitely extended in both senses, and let x denote the ratio $OP : OA$, x being reckoned positive or negative according as $\overline{OP}$ has the same sense with $\overline{OA}$ or the contrary;



then $\overline{OP} = x\alpha$.

Let P be any point in the plane of two given vectors $\overline{OA}$, $\overline{OB}$, and let PM, PN be drawn from P parallel respectively to OB, OA to meet OA, OB in M, N, and let $\overline{OA}$, $\overline{OB}$ be denoted respectively by α, β : then if $\overline{OM} = x\alpha$ and $\overline{ON} = y\beta$,



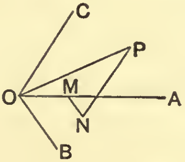


$$\overline{OP} = \overline{OM} + \overline{MP} = \overline{OM} + \overline{ON} = x\alpha + y\beta.$$

Hence the position of any point P in a plane may be defined by the vector drawn to it from the fixed point O, expressed in terms of two given vectors $\overline{OA}$, $\overline{OB}$ (or α, β) by means of the two ratios or scalars x, y reckoned positive or negative according as $\overline{OM}$, $\overline{ON}$ have or have not the same senses respectively as $\overline{OA}$, $\overline{OB}$.

It is plain that for a given point P there is one and only one value of x and y, and that for given values of x, y there is one and only one point : hence, if $x\alpha + y\beta = x'\alpha + y'\beta$, it follows that $x = x'$ and $y = y'$.

Lastly, let $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ be three vectors not in the same plane denoted respectively by α, β, γ, and let P be any point in space : let PN, parallel to OC, meet the plane OAB in N, and NM, parallel to OB, meet OA in M, and let $\overline{OM} = x\alpha$, $\overline{MN} = y\beta$, $\overline{NP} = z\gamma$, then $\overline{OP} = \overline{OM} + \overline{MN} + \overline{NP} = x\alpha + y\beta + z\gamma$.



Thus the position of any point P in space may be defined by the vector drawn to it from the fixed point O, expressed in terms of three given vectors (not co-planar) $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ (or α, β, γ), by means of the three ratios or scalars x, y, z, reckoned positive or negative according as $\overline{OM}$, $\overline{MN}$, $\overline{NP}$ have or have not the same senses respectively as $\overline{OA}$. $\overline{OB}$, $\overline{OC}$.

For a given point P there is one and only one value of each of the scalars x, y, z, and for given values of x, y, z there is one and only one point P: hence, if $x\alpha + y\beta + z\gamma = x'\alpha + y'\beta + z'\gamma$ (α, β, γ not being co-planar), $x = x'$, $y = y'$, $z = z'$.

8. *Illustrative examples.*

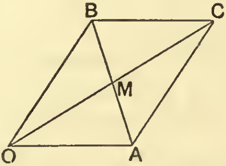
In the examples which follow the vectors of the points A, B, C to an origin O or $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ are denoted by α, β, γ respectively.

If the origin be changed to O', and $\overline{OO'}$ be denoted by ω, and $\overline{O'A}$, $\overline{O'B}$, $\overline{O'C}$ by α', β', γ', it is evident that

$$\alpha = \omega + \alpha', \ \beta = \omega + \beta', \ \gamma = \omega + \gamma'.$$

1°. To find the vector of the middle point (M) of AB.

In this case $\overline{MB} = \overline{AM} = -\overline{MA}$, or $\overline{MA} + \overline{MB} = O$;



but $\qquad \overline{OM} + \overline{MA} = \overline{OA}$ and $\overline{OM} + \overline{MB} = \overline{OB}$,

therefore by addition, $2\overline{OM} + \overline{MA} + \overline{MB} = \overline{OA} + \overline{OB}$

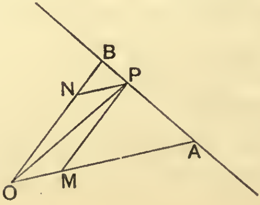
and $\qquad 2\overline{OM} = \overline{OA} + \overline{OB}$,

whence $\qquad \overline{OM} = \frac{1}{2}(\alpha + \beta)$.

If the parallelogram $OACB$ be completed, $\overline{OC} = \alpha + \beta$, therefore $\overline{OM} = \frac{1}{2}\overline{OC}$, so that the middle point of AB is also the middle point of OC. In other words, the diagonals of a parallelogram bisect one another.

2°. To find the vector of any point P in the line AB in terms of the vectors of A and B.

Let the point P be defined as that which divides the line AB in the ratio $y : x$, so that $AP : PB = y : x$ or $x \, . \, AP = y \, . \, PB$, whence $AP = \dfrac{y}{x+y} \, . \, AB$ and $PB = \dfrac{x}{x+y} \, . \, AB$.



Then $\overline{OP} = \overline{OA} + \overline{AP} = \overline{OA} + \frac{y}{x+y}\overline{AB} = \frac{x\overline{OA} + y(\overline{OA} + \overline{AB})}{x+y}$ $= \frac{x\overline{OA} + y\overline{OB}}{x+y}$, or, denoting $\overline{OP}$ by ρ, $\rho = \frac{x\alpha + y\beta}{x+y}$.

This expression leads to the following observations.

a. The form of this expression for the vector of P is independent of the origin assumed.

For if O' be a new origin, and $\overline{OO'} = \omega$, $\overline{O'A} = \alpha'$, $\overline{O'B} = \beta'$, $\overline{O'P} = \rho'$,

$$\text{then } \alpha = \omega + \alpha',\ \beta = \omega + \beta',\ \rho = \omega + \rho',$$

$$\text{and } \frac{x\alpha + y\beta}{x+y} = \frac{x(\omega + \alpha') + y(\omega + \beta')}{x+y} = \omega + \frac{x\alpha' + y\beta'}{x+y},$$

$$\text{whence } \rho' = \frac{x\alpha' + y\beta'}{x+y}.$$

β. If P be between A and B, x, y are both positive: if it be in AB produced, y is positive and x negative: and if it be in BA produced, x is positive and y negative. In all cases $x + y$ is positive, as we are at liberty to assume, since only the ratio and not the absolute values of x, y are involved. Also the expression for ρ depends, not on the absolute values of x and y, but only on their ratio, so that it really involves only *one* arbitrary element, corresponding to the fact that there is only *one* arbitrary element in the position of P, its distance (positive or negative) from *either A or B*.

γ. If we take $\frac{x}{x+y} = m$, then $\frac{y}{x+y} = 1 - m$, and the expression for ρ becomes $\rho = m\alpha + (1-m)\beta$, involving explicitly only the single arbitrary element m.

By reason of the symmetry, however, the form involving both x and y will be found generally the more elegant and convenient.

δ. If α, β, γ be the vectors of three points A, B, C connected by the relation $x\alpha + y\beta + z\gamma = 0$, the condition that they are collinear is $x + y + z = 0$.

For then $z = -(x+y)$, and $\gamma = -\frac{x\alpha + y\beta}{z} = \frac{x\alpha + y\beta}{x+y}$.

ϵ. If PM, PN be drawn to OA, OB, parallel to OB, OA respectively, it follows, from the expression for $\overline{OP}$, that $\overline{OM} = \frac{x}{x+y}\alpha$, $\overline{ON} = \frac{y}{x+y}\beta$, whence $OM : OA = x : x+y = BP : BA$, and $ON : OB = y : x+y = AP : AB$. This is a vector proof of Euclid, vi. 2.

3°. The *mean* point of any number of points.

Let the vectors to the origin O of the n points A, B, C . . . be α, β, γ . . ., then $\frac{\alpha+\beta+\gamma\ldots\ldots}{n}$, or the mean of these vectors, is the vector of a point, which depends only on the relative positions of the points themselves, and not on the origin. It is termed the *mean* point of the n points.

To prove this, we have only to observe that if another origin O' be chosen so that $\overline{OO'} = \omega$, and α', β', γ' . . . be the vectors to the new origin, $\alpha = \omega + \alpha'$, $\beta = \omega + \beta'$ . . ., and therefore

$$\frac{\alpha+\beta+\gamma+\ldots}{n} = \frac{n\omega+\alpha'+\beta'+\gamma'+\ldots}{n} = \omega + \frac{\alpha'+\beta'+\gamma'+\ldots}{n},$$

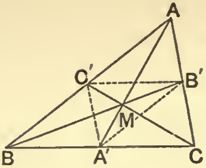
so that the same point is determined by

$$\frac{\alpha+\beta+\gamma+\ldots}{n} \text{ and by } \frac{\alpha'+\beta'+\gamma'+\ldots}{n}.$$

4°. The mean point of three points, A, B, C.

Let μ be the vector of the mean point M: then $\mu = \frac{\alpha+\beta+\gamma}{3}$, which may be expressed in either of the forms

$$\frac{\alpha + 2\,.\,\frac{\beta+\gamma}{2}}{1+2},\quad \frac{\beta + 2\,.\,\frac{\gamma+\alpha}{2}}{1+2},\quad \frac{\gamma + 2\,.\,\frac{\alpha+\beta}{2}}{1+2}.$$



Now $\frac{\beta+\gamma}{2}, \frac{\gamma+\alpha}{2}, \frac{\alpha+\beta}{2}$ are the vectors of the middle points A', B', C' of BC, CA, AB respectively, and the forms above show, by the form for ρ in 2°, that M is a point in each of the lines AA', BB', CC', dividing them in the ratio $2:1$. Hence the "medians" (AA', BB', CC') of a triangle (ABC) meet in a point, the "centroid" (or mean point of the points A, B, C).

Again, since $\mu = \frac{1}{3}\left\{\frac{\beta+\gamma}{2} + \frac{\gamma+\alpha}{2} + \frac{\alpha+\beta}{2}\right\}$, M is also the mean point of A', B', C': or, the centroid of ABC is also that of $A'B'C'$. Also the vector of the middle point of AA' is $\frac{1}{2}\left(\alpha + \frac{\beta+\gamma}{2}\right)$, which is equal to $\frac{1}{2}\left(\frac{\alpha+\beta}{2} + \frac{\alpha+\gamma}{2}\right)$, the vector of the middle point of $B'C'$. Hence AA', $B'C'$ bisect one another, and $AB'A'C'$ is a parallelogram.

It may be left to the student to prove that

$$\overline{AM} + \overline{BM} + \overline{CM} = 0, \text{ and } \overline{A'M} + \overline{B'M} + \overline{C'M} = 0,$$

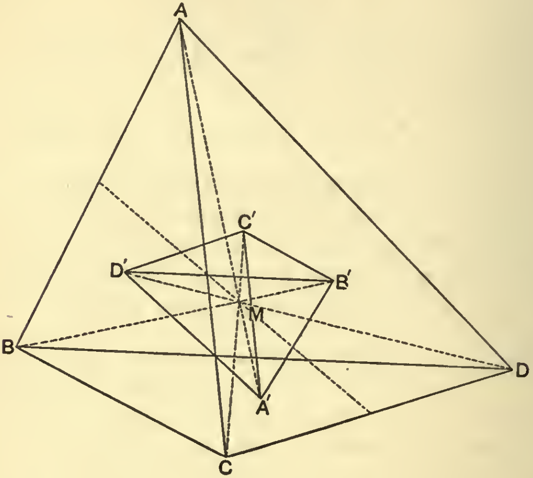
also that $$\overline{B'C'} = \tfrac{1}{2}\overline{CB}, \quad \overline{C'A'} = \tfrac{1}{2}\overline{AC}, \quad \overline{A'B'} = \tfrac{1}{2}\overline{BA}.$$

5°. The mean point of four points, A, B, C, D.

Reasoning as in the foregoing case,

$$\text{since } \mu = \frac{\alpha+\beta+\gamma+\delta}{4} = \frac{\alpha + 3 \cdot \dfrac{\beta+\gamma+\delta}{3}}{1+3},$$

M is a point on the line joining A to the mean point (A') of B, C, D, and dividing it in the ratio of $3:1$. Hence the lines joining the vertices of a tetrahedron to the centroids of the opposite faces all pass through the mean point of the vertices.



Also
$$\mu = \tfrac{1}{2}\left(\frac{\alpha+\beta}{2} + \frac{\gamma+\delta}{2}\right),$$

and therefore M is the middle point of the line joining the middle points of the opposite edges AB, CD.

Hence the lines joining the middle points of each pair of opposite edges of a tetrahedron all pass through the mean point of its vertices and are bisected in that point.

It may also easily be proved that, if A', B', C', D' be the centroids of the faces opposite to A, B, C, D respectively,

(1) M is also the mean point of A', B', C', D'.

(2) The middle point of AA' is the mean point of A, B', C', D'.

(3) The mean point of B', C', D' divides AA' in the ratio of $2 : 1$.

(4) $\overline{B'C'} = \frac{1}{3}\overline{CB}$, $\overline{C'A'} = \frac{1}{3}\overline{AC}$, &c.

(5) The middle points of $A'B'$, $C'D'$ divide the line joining the middle points of AB, CD in the ratios of $2:1$ and $1:2$.

6°. Of the n points $A, B, C, D \ldots$ suppose that x coincide with A, y with B, z with $C \ldots$: then $x+y+z+\ldots=n$, and the vector of the mean point becomes $\dfrac{x\alpha+y\beta+z\gamma+\ldots}{x+y+z+\ldots}$.

In such a case it will be convenient to call $x, y, z \ldots$ the *weights* of the points $A, B, C \ldots$ respectively.

Thus if there are two points A, B having weights x, y respectively, the vector of their mean point is $\dfrac{x\alpha+y\beta}{x+y}$, which, as we have seen, denotes a point P in the line AB, dividing it so that $AP:PB=y:x$. Thus any point P in the line through A and B may be regarded as the mean point of A, B by assigning suitable weights (positive or negative) to these points.

7°. Let the three points A, B, C have weights x, y, z respectively: then ρ being the vector of their mean point P,

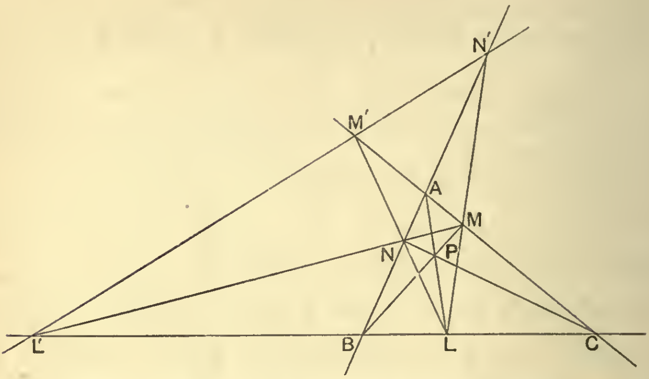
$$\rho=\frac{x\alpha+y\beta+z\gamma}{x+y+z}.$$

a. The expression for ρ may be written in the form

$$\frac{x\alpha+(y+z)\dfrac{y\beta+z\gamma}{y+z}}{x+(y+z)}.$$

Hence, if L be the point in BC which divides BC so that $\overline{BL}:\overline{LC}=z:y$, P is a point in the plane ABC and in the line AL, which divides AL so that $\overline{AP}:\overline{PL}=y+z:x$.

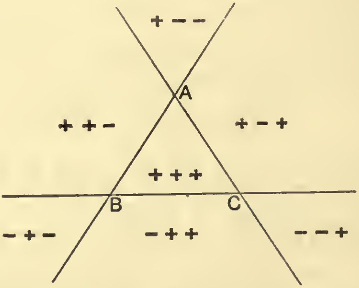
In like manner it may be shown that, if $\overline{CM}:\overline{MA}=x:z$, P is a point in BM which divides it so that $\overline{BP}:\overline{PM}=z+x:y$, and that if $\overline{AN}:\overline{NB}=y:x$, P is a point in CN which divides it so that $\overline{CP}:\overline{PN}=x+y:z$.



It is plain that by assigning suitable values (positive or negative) to x, y, z, P may be any point in the plane ABC, so that the vector of any point in the plane ABC may be expressed in the form $\dfrac{x\alpha + y\beta + z\gamma}{x+y+z}$.

Hence, if $x\alpha + y\beta + z\gamma + u\delta = 0$ be the relation connecting the vectors of four points A, B, C, D or determining either in terms of the other three, the condition that they should be co-planar is

$$x + y + z + u = 0.$$



It should be observed* that, assuming $x+y+z$ to be positive, if x, y, z are all positive, P lies within the triangle ABC: and that the other possible permutations of signs correspond to the six regions outside the triangle, as shown in the annexed diagram, in which the signs are written in the order of those of x, y, z respectively.

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* On a spherical surface three great circles divide the surface into *eight* regions, which may be distinguished by signs as above for a plane with − − −

β. The ratio compounded of the ratios $\overline{BL} : \overline{LC}$, $\overline{CM} : \overline{MA}$, $\overline{AN} : \overline{NB}$ is $xyz : xyz$ or 1.

Hence the theorem, known as Ceva's Theorem, that if lines drawn from A, B, C through a point P meet the opposite sides (or these produced) of the triangle ABC in L, M, N respectively, $\dfrac{\overline{BL}}{\overline{LC}} \cdot \dfrac{\overline{CM}}{\overline{MA}} \cdot \dfrac{\overline{AN}}{\overline{NB}} = 1$: or,* as it is usually expressed in geometrical treatises, $BL \,.\, CM \,.\, AN = LC \,.\, MA \,.\, NB$.

γ. Let MN meet BC in L'. Then, since the vectors of M and N are $\dfrac{z\gamma + x\alpha}{z+x}$ and $\dfrac{x\alpha + y\beta}{x+y}$ respectively, the vector of any point in MN may be expressed as $\dfrac{p}{p+q} \cdot \dfrac{z\gamma + x\alpha}{z+x} + \dfrac{q}{p+q}\,\dfrac{x\alpha + y\beta}{x+y}$, and there must be some value of p/q for which this point is L'. Now α will disappear from this expression if $\dfrac{p}{z+x} + \dfrac{q}{x+y} = 0$,

so that
$$-\frac{p}{z+x} = \frac{q}{x+y} = \frac{p+q}{y-z},$$

in which case the expression reduces to

$$-\frac{1}{y-z}\,(z\gamma + x\alpha) + \frac{1}{y-z}\,(x\alpha + y\beta) \text{ or } \frac{y\beta - z\gamma}{y-z},$$

which is the vector of a point in BC and therefore the vector of L'. Hence L' is such that $\overline{L'B} : \overline{L'C} = -z : y$ and $\overline{L'M} : \overline{L'N} = q : p = -(x+y) : (z+x)$. Similarly if NL, LM meet CA, AB

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for the eighth region. If the surface becomes a plane by the radius becoming infinite, one of these regions goes off to infinity.

* The multiplication of vectors not having been yet considered, we are not entitled to write $\overline{BL} \,.\, \overline{CM} \,.\, \overline{AN} = \overline{LC} \,.\, \overline{MA} \,.\, \overline{NB}$.

respectively in M', N', the vector of M' is $\frac{z\gamma - x\alpha}{z-x}$ and that of N' $\frac{x\alpha - y\beta}{x-y}$, and $\overline{CM'} : \overline{M'A} = -x : z$, $AN' : NB = -y : x$.

Hence if λ', μ', ν' denote the vectors of L', M', N'

$$\lambda' = \frac{y\beta - z\gamma}{y-z} \quad \mu' = \frac{z\gamma - x\alpha}{z-x} \quad \nu' = \frac{x\alpha - y\beta}{x-y}$$

and therefore $\quad (y-z)\,\lambda' + (z-x)\,\mu' + (x-y)\,\nu' = 0,$
whence (by 2° δ) L', M', N' are collinear,
since $\quad (y-z) + (z-x) + (x-y) = 0.$

δ. The line L', M', N' has been derived from the point P, for which the ratios $x : y : z$ have definite values. It is plain that the steps might be reversed, so that starting from any line L', M', N' definite values of the ratios $x : y : z$ would be obtained, and thence definite points L, M, N and a definite point P, whose vector is $\frac{x\alpha + y\beta + z\gamma}{x+y+z}$. Hence between the point P and the line L', M', N' there is this correspondence that, given either, the other is determined with reference to the triangle ABC.

The ratio compounded of the ratios $\overline{BL'} : \overline{L'C}$, $\overline{CM'} : \overline{M'A}$, $\overline{AN'} : \overline{N'B}$ is $-zxy : yzx$ or -1, so that $\frac{\overline{BL'}}{\overline{L'C}} \cdot \frac{\overline{CM'}}{\overline{M'A}} \cdot \frac{\overline{AN'}}{\overline{N'B}} = -1$. Hence the theorem, known as the Theorem of Menelaus, that if a transversal meet the sides BC, CA, AB of the triangle ABC in the points L', M', N' respectively, $\frac{\overline{BL'}}{\overline{L'C}} \cdot \frac{\overline{CM'}}{\overline{M'A}} \cdot \frac{\overline{AN'}}{\overline{N'B}} = -1$ and the converse.

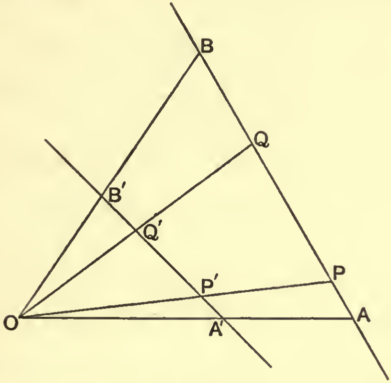
8°. Let P, Q be two points in the line AB, such that, $\overline{OA}$, $\overline{OB}$ being denoted by α, β,

$$\overline{OP} = \frac{x\alpha + y\beta}{x+y}, \; \overline{OQ} = \frac{x'\alpha + y'\beta}{x'+y'}.$$

Then

$$\frac{\overline{AP}}{\overline{PB}} = \frac{y}{x} \text{ and } \frac{\overline{AQ}}{\overline{QB}} = \frac{y'}{x'},$$

and therefore the (*anharmonic* or) *cross ratio* of the range A, P, Q, B, which is $\dfrac{\overline{AP}}{\overline{BP}}\Big/\dfrac{\overline{AQ}}{\overline{BQ}}$, is $\dfrac{x'y}{xy'}$.



Let another transversal meet the pencil $O(APQB)$ in A', P', Q', B', then if $\overline{OA'} = m\,.\,\overline{OA}$, $\overline{OB'} = n\,.\,\overline{OB}$, $\overline{OP'} = p\,.\,\overline{OP}$,

$$\overline{OP'} = p\,.\,\frac{x\alpha + y\beta}{x+y} = p\,.\,\frac{\dfrac{x}{m}+\dfrac{y}{n}}{x+y}\,.\,\frac{\dfrac{x}{m}\,.\,m\alpha + \dfrac{y}{n}\,.\,n\beta}{\dfrac{x}{m}+\dfrac{y}{n}} = p\,.\,\frac{\dfrac{x}{m}+\dfrac{y}{n}}{x+y}\,.\,\overline{OP},$$

since the last factor is the vector of some point in $A'B'$ and therefore of P', whence

$$\frac{\overline{OP'}}{\overline{OP}} \text{ or } p = \frac{x+y}{\dfrac{x}{m}+\dfrac{y}{n}}, \text{ and } \overline{OP'} = \frac{x\alpha + y\beta}{\dfrac{x}{m}+\dfrac{y}{n}}.$$

Similarly $$\frac{\overline{OQ'}}{\overline{OQ}} \text{ or } q = \frac{x'+y'}{\dfrac{x'}{m}+\dfrac{y'}{n}}, \text{ and } \overline{OQ'} = \frac{x'\alpha + y'\beta}{\dfrac{x'}{m}+\dfrac{y'}{n}}.$$

Hence $$\frac{\overline{A'P'}}{\overline{B'P'}} = \frac{y}{n}\,.\,\frac{m}{x} \text{ and } \frac{\overline{A'Q'}}{\overline{B'Q'}} = \frac{y'}{n}\,.\,\frac{m}{x'},$$

so that $$\frac{\overline{A'P'}}{\overline{B'P'}}\bigg/\frac{\overline{A'Q'}}{\overline{B'Q'}} = \frac{x'y}{xy'} = \frac{\overline{AP}}{\overline{BP}}\bigg/\frac{\overline{AQ}}{\overline{BQ}},$$

or *the cross ratio of the range $A'P'Q'B'$ is the same as that of the range $APQB$.*

Hence the same ratio is said to be also the *cross ratio of the pencil $O\,(APQB)$.*

If $\dfrac{x'}{y'} = -\dfrac{x}{y}$ or $x'y + xy' = 0$, the cross ratio $= -1$, and P, Q divide AB externally and internally in the same ratio, or *harmonically.*

Hence if P, Q be points which divide AB harmonically,

$$\overline{OP} = \frac{x\alpha + y\beta}{x+y},\ \overline{OQ} = \frac{x\alpha - y\beta}{x-y}.$$

The development of the geometrical consequences of these results is beyond the scope of the present work.

EXAMPLES.

1. Verify geometrically the formulae

$$\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} = \alpha,\ \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2} = \beta.$$

2. The line passing through the middle points of AB, AC bisects every line drawn from A to BC, and therefore is parallel to BC.

Also verify Euc. VI. 2 generally by vectors.

3. If the vectors $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ are denoted by α, β, γ respectively, and I is the centre of the inscribed circle of the triangle ABC, $\overline{OI} = \dfrac{a\alpha + b\beta + c\gamma}{a+b+c}$, where a, b, c are the lengths of BC, CA, AB respectively.

4. If I_1, I_2, I_3 are the centres of the escribed circles, express $\overline{OI_1}$, $\overline{OI_2}$, $\overline{OI_3}$ in terms of α, β, γ, and prove that

$$s\,.\,\overline{OI} = (s-a)\,.\,\overline{OI}_1 + (s-b)\,\overline{OI}_2 + (s-c)\,\overline{OI}_3, \text{ where } s = \frac{a+b+c}{2}.$$

5. If the inscribed circle touches BC, CA, AB in D, E, F respectively, find the vectors $\overline{OD}$, $\overline{OE}$, $\overline{OF}$ and prove that

$$a \, . \, \overline{ID} + b \, . \, \overline{IE} + c \, . \, \overline{IF} = 0.$$

Also that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ meet in a point K, such that

$$\overline{OK} = \frac{r_1\alpha + r_2\beta + r_3\gamma}{r_1 + r_2 + r_3},$$

where r_1, r_2, r_3 are the respective radii of the escribed circles.

6. Find the points where the line through I and M (the mean point of A, B, C) meets BC, CA, AB respectively, and shew that if $b + c = 2a$, IM is parallel to BC.

7. If a straight line meets BC, CA, AB in D, E, F respectively, the middle points of AD, BE, CF are collinear.

8. If N is a point in the median AM of the triangle ABC such that $MN : NA :: l : m$, and if BN meets CA in P, then $CP : PA :: 2l : m$. If also CN meets AB in Q, PQ is parallel to BC.

9. If $\dfrac{x\alpha + y\beta + z\gamma}{x + y + z}$ is the vector of a point in the plane ABC, and that of its isogonal conjugate with respect to the triangle ABC is $\dfrac{x'\alpha + y'\beta + z'\gamma}{x' + y' + z'}$, then $\dfrac{xx'}{a^2} = \dfrac{yy'}{b^2} = \dfrac{zz'}{c^2}$.

10. If A, B, C, D be four points having equal weights, and A', B', C', D' are the mean points of the groups BCD, CDA, DAB, ABC, prove that

(α) The mean point of A', B', C', D' coincides with that of A, B, C, D.

(β) The mean point of B', C', D' divides AA' in the ratio of $2 : 1$.

(γ) $B'C'$ is parallel to BC, $C'A'$ to CA, &c.

(δ) The mean point of A, B', C', D' is the middle point of AA'.

11. Investigate corresponding properties, when A, B, C, D have unequal weights.

12. If α, β, γ are three edges of a parallelepiped meeting in one of its corners, express the diagonals in terms of α, β, γ: and thence shew that they meet and are bisected in one point. Also that the lines joining the middle points of opposite edges are bisected in the same point.

13. If A', B', C' are points in OA, OB, OC respectively, and BC, CA, AB meet $B'C'$, $C'A'$, $A'B'$ respectively in D, E, F, prove that D, E, F are collinear. Also if BC', $B'C$ intersect in D', CA', $C'A$ in E', AB', $A'B$ in F', the planes ABC, $A'B'C'$, $D'E'F'$ all intersect in the line DEF. Also AD', BE', CF' meet in a point (G), and aD', bE', cF' in a point (g), and O, G, g are collinear.

14. If a system of points, of equal or unequal (but fixed) weights, is displaced in any manner, the displacement of its mean point is the vector mean of the displacements of its several points.

APPENDIX ON VECTOR MAGNITUDES.

1. IT has been observed in the Introduction (p. 8) that Magnitudes, which may be made the subject of mathematical treatment, are of various kinds. Some, such as Population, Mass, Energy, Heat, &c., of which the negative is inconceivable, are *purely quantitative.* Others,—such as length measured on a given line in one sense (forwards) of which the negative is length measured in the opposite sense (backwards), time *after* a given epoch of which the negative is time *before* the same, *debit* of which the negative is *credit*, *gain* of which the negative is *loss*, and so on,—are *scalar*, or can be represented by the positive and negative quantities of Ordinary Algebra. There are others, however, which, like the vectors we have considered in this chapter, require for their *complete* definition the *two* elements, *quantity* and *direction*. Such magnitudes can be completely represented by vector lines, which have the same directions and senses as the magnitudes themselves, and lengths (or

tensors) proportional to their *quantities.* These may be properly termed *Vector Magnitudes,* if it can be shown that they can be combined according to the laws of Vector Aggregation. Again there are other Magnitudes, which, besides the elements of a Vector Magnitude, require for their complete specification additional elements as, for instance, forces acting on an extended body or system, (in which case besides quantity and direction, the particular line of action must be given), (Clifford's *rotors*): or screws, wrenches, &c., which have *pitch* as well as quantity, direction, and position (Clifford's *motors*).

Corresponding to these several kinds of magnitudes are the fundamental subdivisions of Algebra in its most general sense. The treatment of purely *quantitative Magnitudes* is contained in Arithmetical Algebra, in which $a-b$, where b is greater than a, is an "impossible" quantity. *Scalar Magnitudes* are the subject of ordinary or Scalar Algebra, in which $\sqrt{-1}$, or such expressions as $a+b\sqrt{-1}$, are "impossible" quantities.

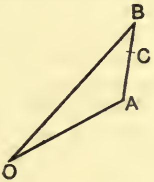
To *Vector Magnitudes,* if they are *limited to one plane,* corresponds Ordinary Algebra without any "impossible" quantities (the "Double Algebra" of De Morgan), as it is the purpose of the present work to show.

For the treatment of Vector Magnitudes in all their generality, and other magnitudes transcending these in complexity, special Algebras are appropriate, as the Quaternions of Hamilton, the Triple Algebra of De Morgan, Biquaternions of Clifford, &c.

We proceed to show that the Elementary Magnitudes of Dynamics, *Velocity, Acceleration, Force* (acting at a given point) are truly *Vector Magnitudes.* For this purpose it is only necessary to prove the following proposition :—

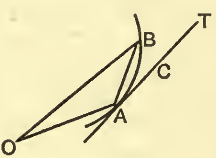
2. *The rate of change of a Vector Magnitude is itself a Vector Magnitude.*

Let $\overline{OA}$ represent at a given instant a Vector Magnitude, which after t units of time becomes changed to $\overline{OB}$, then by the law of Vector Aggregation $\overline{AB}$ represents the total change of the Vector Magnitude in



the interval of t units of time. If the change takes place uniformly, that is, by equal steps in the same constant direction for any equal parts of the interval, the extremity of the vector always lies on the line AB, and moves along that line with uniform speed, so that the change that would be effected in the unit of time is completely represented by the vector $\overline{AC}$, if $\overline{AC} = \frac{1}{t} \times \overline{AB}$. In other words the vector $\overline{AC}$, or $\frac{\overline{AB}}{t}$ (where t is, it will be observed, a pure number), represents the *uniform rate of change* of the vector magnitude at any instant during the interval.

If the change takes place not uniformly, but so that either the direction of the steps into which the total change can be subdivided or the length of the steps for a given fraction of the interval or both are variable, the vector $\frac{\overline{AB}}{t}$ represents the *mean* rate of change during the interval. The extremity of the vector then moves with variable speed

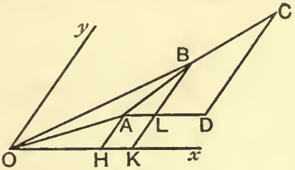


along a curve line from A to B, and as the time t is taken smaller and smaller, so that the point B gets nearer and nearer to A, the mean *speed* $\frac{AB}{t}$ approaches continually a certain limit v, which we may call the *speed* at A of the extremity of the vector, and the direction of $\overline{AB}$ approaches a limiting direction, that of the tangent to the curve at A. Hence if AC be taken along the tangent AT at A, equal to v, the limit of $\frac{AB}{t}$ when t is diminished without limit, $\overline{AC}$ will represent the *rate of change* of the vector at the instant when it coincides with $\overline{OA}$.

Besides thus estimating the *complete* rate of change of a vector as a single vector having a *quantitative*, and also a *directive* element, it may be estimated as the sum of two vectors having *given* directions and therefore each specified by a single (*quantitative*) element. Confining ourselves to vectors in one plane, let Ox, Oy be two given directions in that plane; then, if AH, BK are drawn parallel to Oy to meet Ox in H, K, and AL parallel to Ox to meet BK in L, the vectors $\overline{AL}$, $\overline{LB}$, represent the changes of the vector magnitude in the directions Ox, Oy respectively, which, combined by the vector law of addition, $\overline{AB} = \overline{AL} + \overline{LB}$, are equivalent to the total change. Then, taking the case of *uniform* rate of change, the complete rate of change $\frac{1}{t} \cdot \overline{AB}$ is made up of the component rates $\frac{1}{t} . \overline{AL}$, $\frac{1}{t} . \overline{LB}$ in the directions Ox, Oy, respectively by the vector law of addition

$$\frac{1}{t} \cdot \overline{AB} = \frac{1}{t} \cdot \overline{AL} + \frac{1}{t} . \overline{LB}.$$

Hence if $\overline{AC} = \frac{1}{t}.\overline{AB}$, and $\overline{AD} = \frac{1}{t}.\overline{AL}$, since $\overline{AC} = \overline{AD} + \overline{DC}$, it follows that $\overline{DC} = \frac{1}{t}.\overline{LB}$, and it may be stated that the rates of change represented by $\overline{AD}$, $\overline{DC}$, combine by the vector law of addition into the rate represented by $\overline{AC}$. In the case of variable rate of change, the same conclusion would be arrived at by taking the limits of $\frac{1}{t}.\overline{AB}$, &c., instead of the actual rates or mean rates, as explained above.



Hence we conclude that Rates of change of Vector Magnitudes are themselves also *Vector Magnitudes.*

3. *Velocity, a Vector Magnitude.*

The *velocity* of a point is the rate of change of a vector defining its position at any instant. Hence, if $\overline{OA}$, $\overline{OB}$, in the preceding section are vectors defining the position of a moving point, which passes from A to B in t units of time, $\overline{AC}$ represents the velocity of the point when at A, and of its two elements the length AC represents the tensor element, termed the *speed* of the point, while its direction is the *direction* of the velocity. Also, since $\overline{AC}$ is the vector sum of $\overline{AD}$ and $\overline{DC}$, the lengths AD, DC, represent two speeds along *given* directions, which may be regarded as *component* velocities, which, combined according to the vector law of addition, give the complete or *resultant* velocity.

This is the first elementary proposition of Kinematics, known as the "Triangle (or Parallelogram) of Velocities."

4. *Acceleration, a Vector Magnitude.*

The *Acceleration* of a point is the rate of change of its velocity.

Hence, if $\overline{OA}$, $\overline{OB}$ are taken to represent the velocities of a point at a given instant and after an interval of t units of time, respectively, by similar reasoning to that regarding velocity, the vector $\overline{AC}$ represents the acceleration of the point at the first instant, and of its two elements the length AC represents the tensor element, while its direction is the direction of the acceleration. Also it is plain in like manner, that the total acceleration, represented by $\overline{AC}$, may be regarded as made up by vector addition of component accelerations, represented by $\overline{AD}$, $\overline{DC}$, which are the *quickenings*, or rates of change of *speed*, in two *given* directions.

Hence the proposition of Kinematics known as "The Triangle (or Parallelogram) of Accelerations."

5. *Force, a Vector Magnitude.*

It follows from the second of Newton's "Laws (or Axioms) of Motion," that forces have the directions of the accelerations which they produce in a particle, and are measured as to quantity by the acceleration multiplied by a certain *purely quantitative* constant belonging to the particle on which the force acts, which is regarded as the measure of the *quantity of matter* the particle contains, and is termed its *Mass*. Hence, the product of a vector by a pure number being a vector having the same direction, what has been seen above to be true of velocities and accelerations must be true also of Forces.

Hence the fundamental proposition of (non-linear) Dynamics, known as the "Triangle (or Parallelogram) of Forces."

6. *Illustrations.*

To develop the consequences of these fundamental propositions is the proper object of special treatises on Kinematics and Dynamics. We may, however, by way of illustration, call attention to the dynamical reading of one or two of the geometrical consequences established in the present Chapter.

Thus, if $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, represent three forces acting on a particle, and G is the mean point of A, B, C, the resultant of these forces is represented by $3.\overline{OG}$, and in like manner if G were the mean point of n such points, the resultant would be represented by $n.\overline{OG}$.

Hence, supposing O to coincide with G, the forces represented by $\overline{GA}$, $\overline{GB}$, . . . form a system in equilibrium, when applied to the same particle.

Again, if O moves off to an infinite distance, the vectors all become parallel, and it follows that the resultant of a system of equal parallel forces acting through the points A, B, C . . . acts in a parallel direction through G, their mean point, which is then known as the *centre* of the system of equal parallel forces. This may easily be generalized to meet the case of unequal parallel forces, and so to that of the weights of the several particles of a system, when the mean point becomes their *centre of gravity*.

Again, if the points of a system are regarded as material points having masses proportional to the (geometrical) weights of the points (§ 8-6°), it may be inferred (as in Ex. 14 above) that the velocity of the centre of mass of a material system is the vector-mean of the velocities of the several particles or bodies of the system, so that this becomes the proper measure of the motion of translation of the system. The same is easily seen to be true of the accelerations. Then the equal and opposite forces of a stress between two particles producing equal and opposite mass-accelerations (or rates of change of momentum), these last may be left out of account in determining the acceleration of the centre of mass, and that acceleration is the vector-mean of the accelerations of the several particles due to the *external* forces only. It may therefore be stated that the motion of translation of a material system, or the motion of its centre of mass, is the same as that of the whole mass supposed collected at its centre of mass and acted on by forces equal to and in the same direction as the actual external forces.

CHAPTER II

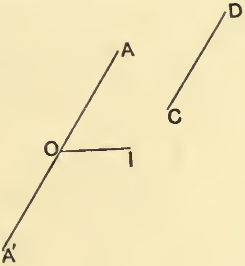
MULTIPLICATION OF COPLANAR VECTORS

1. It has been shown in the preceding chapter that the laws of *aggregation* of vectors are in accordance with the fundamental laws of ordinary algebra, and this without any limitation as to the vectors themselves.

In the present chapter we proceed to show that for vectors limited to lie in *the same plane*, or *coplanar* vectors, a natural extension of the term "multiplication" leads to laws also identical with those of ordinary algebra. The consideration of the products and quotients of vectors generally in space of three dimensions leads to the higher algebra of Sir W. Hamilton's Quaternions or De Morgan's Triple Algebra, the Ausdehnungs Lehre of Grassmann, &c., with laws divergent from those of ordinary algebra, with which alone the present treatise is concerned.

2. *Relation of vectors in a given plane to the prime vector.*

Let a definite vector $(\overline{OI})$ be taken, whose length is the unit of length, and let this be called the prime vector and denoted by 1. Then any other vector in the plane may be defined by the ratio of its length to the length of the prime vector and its inclination to the same. This ratio, a *purely quantitative* (*non-scalar* or *signless*) quantity, we shall call (after Hamilton) the *tensor* of the vector. The inclination is measured by the angle between the prime vector $\overline{OI}$ and a vector drawn from O *equal* (c. i, § 3) to the given vector: this angle being the angle turned through by a line revolving about O from coincidence with OI, till it coincides with the vector so drawn, and reckoned positive or negative according as the rotation is one or other of the two senses of rotation in the plane, known as *clockwise*, or the sense of rotation of the hands of a clock looked at from the front, and *counter-clockwise*, the contrary sense. In this treatise the latter (the *counter-clockwise*) sense is regarded as positive, and the former as negative, and all the figures are drawn accordingly. For the present too the right angle, which we shall denote by the symbol $\llcorner$, will be taken as the unit angle, so that if u is the measure of an angle, $90\,u$ is its measure in degrees: or $u^{\llcorner} = 90u^{\circ}$. Hence a vector, whose tensor is a and inclination $\alpha^{\llcorner}$, may be formed from the prime vector $\overline{OI}$ by taking a length along $\overline{OI}$ which shall have to OI the ratio of a to 1 and turning it about O through the angle $\alpha^{\llcorner}$. This vector we shall represent by $\overline{OA}$ and denote it for the present by (a, α).



Thus if CD be a line equal and parallel to OA and in the same sense, $\overline{CD} = \overline{OA} = (a, \alpha)$: and, if OA' be equal to OA in AO produced, the inclination of $\overline{OA'}$ being $\overline{\alpha + 2}^{\llcorner}$, $\overline{DC} = \overline{OA'} = (a, \alpha + 2)$ so that, since $\overline{CD} + \overline{DC} = 0$,

$$(a, \alpha) + (a, \alpha + 2) = 0, \text{ or } (a, \alpha + 2) = -(a, \alpha):$$

or, the addition of two right angles to the inclination of a vector is equivalent to the reversal of its algebraical sign. Also, since the rotation of $\overline{OA}$ through four right angles or any multiple (positive or negative) of four right angles brings it back to its original position; if n be any positive or negative integer,

$$(a, \alpha+4n)=(a, \alpha) \text{ and } (a, \alpha+4n+2)=-(a, \alpha).$$

It should be observed that, having regard to the algebraical *sign* of an angle as denoting the *sense* in which it is measured, if AOB denote the angle which OB makes with OA, BOA will denote the angle in the contrary sense which OA makes with OB, and consequently that

$$BOA=-AOB, \text{ or } AOB+BOA=0.$$

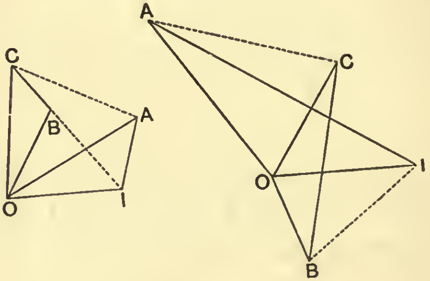
Also that, whatever be the relative positions of OA, OB, OC,

$$AOC=AOB+BOC, \text{ or } AOB+BOC+COA=0.$$

3. *Vector multiplication.*

Multiplication has been defined (Introd., § 5) as the operation of "doing with the multiplicand what must be done with the unit to form the multiplier."

Now to form the vector (a, α) the prime vector $(1, 0)$ has its tensor altered in the ratio a to 1, and its direction changed by rotation through the angle α: hence if the vector (b, β) is



multiplied by (a, α), its tensor b must be changed to ab, and its direction changed by rotating through the angle $\alpha\llcorner$, so that its new inclination is $(\alpha+\beta)$. We may therefore write,

$$(a, \alpha)\times(b, \beta)=(ab, \alpha+\beta),$$

regarding the factor before the sign of multiplication as the multiplier.

To give geometrical expression to the foregoing, let $\overline{OA}$, $\overline{OB}$ be the vectors (a, α) and (b, β) respectively, then if on OB a triangle OBC be described equiangular to OIA, the side OB corresponding to OI and the *angle BOC having the same sense as the angle IOA*, $\overline{OC}$ will be the vector $(ab, \alpha+\beta)$, or the product of $\overline{OB}$ multiplied by $\overline{OA}$. For, since the triangles are similar, $OC : OB = OA : OI$, or $OC : b = a : 1$, or $OC = ab$, and the angle $IOC = IOB + BOC = IOA + IOB = (\alpha+\beta)^{\llcorner}$.

We have therefore $\overline{OA} \times \overline{OB} = \overline{OC}$.

4. *Commutative law of multiplication.*

We have seen that, regarding (a, α) as the multiplier,

$$(a, \alpha) \times (b, \beta) = (ab, \alpha+\beta).$$

In like manner, regarding (b, β) as the multiplier and (a, α) as the multiplicand,

$$(b, \beta) \times (a, \alpha) = (ba, \beta+\alpha),$$

but since the commutative law holds both for sums and products of numerical and scalar magnitudes,

so that $$ba = ab \text{ and } \beta+\alpha = \alpha+\beta,$$

$$(ba, \beta+\alpha) = (ab, \alpha+\beta),$$

and therefore $$(a, \alpha) \times (b, \beta) = (b, \beta) \times (a, \alpha).$$

Hence the commutative law is true for the multiplication of vectors.

The same may readily be proved geometrically.

For IB and AC being joined,

since $$OC : OB :: OA : OI,$$

therefore alternately $$OC : OA :: OB : OI,$$

and the angle $AOC = AOB + BOC = AOB + IOA = IOB$, whence the triangle AOC is similar to the triangle IOB: so that OC will be obtained either by describing on OB a triangle similar to IOA, or by describing on OA a triangle similar to IOB.

Hence
$$\overline{OA} \times \overline{OB} = \overline{OB} \times \overline{OA}.$$

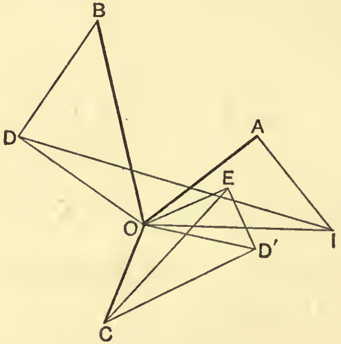
5. *Associative law of multiplication.*

Since
$$(a, \alpha) \times (b, \beta) = (ab, \alpha + \beta)$$
$$\{(a, \alpha) \times (b, \beta)\} \times (c, \gamma) = (ab, \alpha + \beta) \times (c, \gamma) = (ab.c, \overline{\alpha + \beta} + \gamma).$$

Also since
$$(b, \beta) \times (c, \gamma) = (bc, \beta + \gamma),$$
$$(a, \alpha) \times \{(b, \beta) \times (c, \gamma)\} = (a, \alpha) \times (bc, \beta + \gamma) = (a.bc, \alpha + \overline{\beta + \gamma}).$$

But, since the associative law holds both for sums and products of numerical and scalar magnitudes so that $ab \times c = a \times bc$ and $\overline{\alpha + \beta} + \gamma = \alpha + \overline{\beta + \gamma}$,
$$(ab.c, \overline{\alpha + \beta} + \gamma) = (a.bc, \alpha + \overline{\beta + \gamma}),$$
and therefore
$$\{(a, \alpha) \times (b, \beta)\} \times (c, \gamma) = (a, \alpha) \times \{(b, \beta) \times (c, \gamma)\}.$$

Hence the associative law is true for the multiplication of vectors.

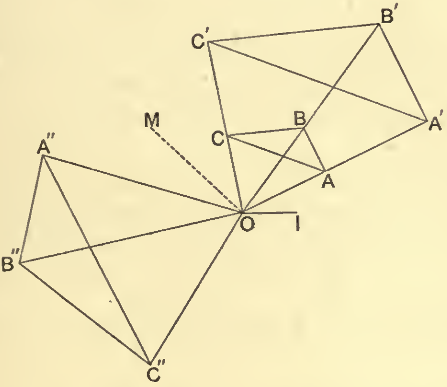


The geometrical verification of the foregoing, the details of which we leave as an exercise for the student, may be obtained from the annexed figure, in which $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ being the three vectors, $\overline{OD}$ is the product of $\overline{OA}$, $\overline{OB}$, and $\overline{OD'}$ that of OB, OC, and $\overline{OE}$ is the product at once of $\overline{OD}$ and $\overline{OC}$, that is, of $(\overline{OA}\,.\,\overline{OB})\,.\,\overline{OC}$ and of $\overline{OA}$ and $\overline{OD'}$, that is, of $\overline{OA}\,.\,(\overline{OB}\,.\,\overline{OC})$.

The student is also recommended to construct his own figure for three vectors drawn at random; or to scale for particular vectors, as $(1, \frac{1}{2}^{\llcorner})$, $(\frac{1}{2}, 3^{\llcorner})$, $(2, -1^{\llcorner})$.

6. *The distributive law.*

Let the vectors (a, α), (b, β) be represented by $\overline{BC}$, $\overline{CA}$ respectively, then their vector-sum (c, γ) will be represented by $\overline{BA}$. Also let any other vector (m, μ) be represented by $\overline{OM}$.



Produce OA, OB, OC to A', B', C' respectively so that

$$OA' = m.OA, \quad OB' = m.OB, \quad OC' = m.OC\,;$$

then it is known from elementary geometry that $B'C'$, $C\,A'$, $A'B$ are respectively parallel to BC, CA, AB and in the same ratio, each to each, as OA' to OA, and hence

$$\overline{B'C'} = m\overline{BC},\ \overline{C'A'} = m\overline{CA},\ \overline{B'A'} = m\overline{BA}.$$

Now $$\overline{B'A'} = \overline{B'C'} + \overline{C'A'}$$

herefore $$m.\overline{BA} = m.\overline{BC} + m\overline{CA},$$

or, since $$\overline{BA} = \overline{BC} + \overline{CA},$$

$$m\ (\overline{BC} + \overline{CA}) = m\overline{BC} + m\overline{CA},$$

that is, $$m\,\{(a,\ \alpha) + (b,\ \beta)\} = m.\ (a,\ \alpha) + m.\ (b,\ \beta).$$

Now let the figure $OA'B'C'$ be turned about O, so that OA' turns through the angle μ or IOM, let its new position be $OA''B''C''$, then all the other lines of the figure have turned through the same angle μ. Hence

$$\overline{B''C''} = (m,\ \mu).\overline{BC},\quad \overline{C''A''} = (m,\ \mu).\overline{CA},\quad \text{and}\quad \overline{B''A''} = (m,\ \mu).\overline{BA},$$

but $$\overline{B''A''} = \overline{B''C''} + \overline{C''A''},$$

therefore $$(m,\ \mu).\ \overline{BA} = (m,\ \mu).\overline{BC} + (m,\ \mu).\overline{CA},$$

or $$(m,\ \mu)\ (\overline{BC} + \overline{CA}) = (m,\ \mu)\ \overline{BC} + (m,\ \mu).\overline{CA},$$

that is, $$(m,\ \mu)\ \{(a,\ \alpha) + (b,\ \beta)\} = (m,\ \mu)\ .\ (a,\ \alpha) + (m,\ \mu)\ .\ (b,\ \beta),$$

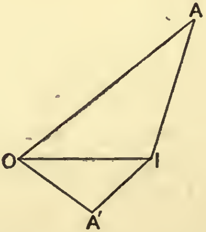
which proves the truth of the distributive law for vectors, with the interpretation of multiplication given above.

7. *Reciprocal of a vector.*

Two vectors are ***reciprocal***, when their product $=1$, so that $(a,\ \alpha)$ and $(b,\ \beta)$ are reciprocal, if $(a,\ \alpha) \times (b,\ \beta) = 1$ or $(1,\ 0)$.

But $$(a,\ \alpha) \times (b,\ \beta) = (ab,\ \alpha + \beta),$$

and therefore $ab = 1,\ \alpha + \beta = 0,$ or $b = \frac{1}{a},\ \beta = -\alpha,$



or the reciprocal of the vector $(a,\ \alpha)$ is the vector $\left(\frac{1}{a},\ -\alpha\right)$. Geometrically, on OI describe the triangle IOA' equiangular to

AOI, the side OI of the former corresponding to OA of the latter, and the angle IOA' being equal to, and having the same sense as, AOI or the opposite sense to IOA, then $\overline{OA'}$ is the reciprocal of $\overline{OA}$. For if $\overline{OA}=(a,\ \alpha)$, since

$$OA' : OI :: OI : OA \quad \text{or} \quad OA' : 1 :: 1 : a,$$

$$OA'=\frac{1}{a} \text{ and } IOA'=-\alpha, \text{ whence } \overline{OA'}=\left(\frac{1}{a},\ -\alpha\right).$$

8. *Vector-division.*

Division is the inverse of multiplication, so that if

$$(a,\ \alpha)\times(b,\ \beta)=(ab,\ \alpha+\beta),$$

$$(ab,\ \alpha+\beta)\div(a,\ \alpha)=(b,\ \beta) \text{ or } \frac{(ab,\ \alpha+\beta)}{(a,\ \alpha)}=(b,\ \beta).$$

Let $ab=c$, and $\alpha+\beta=\gamma$, then $b=\frac{c}{a}$ and $\beta=\gamma-\alpha$, so that

$$\frac{(c,\ \gamma)}{(a,\ \alpha)}=\left(\frac{c}{a},\ \gamma-\alpha\right).$$

Hence the quotient of one vector divided by another is the vector, whose tensor is the quotient of the tensors, and inclination the difference of the inclinations or, more definitely, the sum of the inclination of the dividend and the inclination, reversed in sense, of the divisor.

Also, since $\left(\frac{c}{a},\ \gamma-\alpha\right)=(c,\ \gamma)\times\left(\frac{1}{a},\ -\alpha\right)$

$$\frac{(c,\ \gamma)}{(a,\ \alpha)}=(c,\ \gamma)\times\frac{1}{(a,\ \alpha)},$$

or the quotient of one vector divided by another is the product of the former and the reciprocal of the latter.

9. *Scalar powers of a vector.*

The interpretation of $(a,\ \alpha)^m$, where m is a scalar, follows immediately from the foregoing articles.

1°. If m is positive and integral,

$$(a,\ \alpha)^m=(a,\ \alpha)\,.\,(a,\ \alpha)\,.\,(a,\ \alpha)\ \ldots\ldots \text{ to } m \text{ factors}$$
$$=(a^m,\ m\alpha) \text{ by the laws of multiplication.}$$

2°. If m is positive and fractional $\left(=\frac{p}{q}\text{ suppose}\right)$,

$$\text{then, since } \left(a^{\frac{p}{q}}, \frac{p}{q}\alpha\right)^q = (a^p, p\alpha) = (a, \alpha)^p,$$

$$\left(a^{\frac{p}{q}}, \frac{p}{q}\alpha\right) = (a, \alpha)^{\frac{p}{q}}.$$

3°. If m is negative, since the rules of indices are to hold good universally,

$$(a, \alpha)^{-m} = \frac{1}{(a, \alpha)^m} = \frac{1}{(a^m, m\alpha)} = \left(\frac{1}{a^m}, -m\alpha\right) = (a^{-m}, -m\alpha).$$

Hence universally for any scalar value of m,

$$(a, \alpha)^m = (a^m, m\alpha),$$

assuming by the law of continuity or by limits that the result is true, when m is incommensurable with unity, no less than when it is commensurable.

10. *Multiplicity of values of scalar powers of a vector.*

Since $(a, \alpha) = (a, \alpha + 4n)$, where n is any positive or negative integer,

$$(a, \alpha)^m = (a^m, m\alpha + 4mn).$$

If m is an integer, $4mn$ is a multiple of 4, and $(a^m, m\alpha + 4mn) = (a^m, m\alpha)$, so that $(a, \alpha)^m$ has only one value.

If m is fractional and $=\frac{p}{q}$, p being prime to q, the form $m\alpha + 4mn$ or $\frac{p}{q}\alpha + \frac{4np}{q}$ will define q different directions, as n receives any q consecutive values in the series

$$\ldots -3, -2, -1, \quad 0, 1, 2, 3 \ldots,$$

since no two values of n differing by an integer less than q can make the two values of $\frac{np}{q}$ differ by an integer. Also it will define no more than q different directions, since the $\overline{q+1}^{\text{th}}$, $\overline{q+2}^{\text{th}}$, &c. values of p, differ from the 1st, 2nd, &c. respectively by a multiple of four right angles; and thus, if the series of values of n is continued indefinitely, the same q directions are repeated over and over again continually, each direction being inclined to the next preceding by the fraction $\frac{p}{q}$ of four right angles. Hence there are q, and only q, different values of $(a, \alpha)^{\frac{p}{q}}$, where p is prime to q, and these have all the same tensor, $\sqrt[q]{a^p}$, and so may be represented by radii of the same circle drawn to q points on the circumference, each in advance of the preceding by p q^{ths} of the circumference. Lastly, if m is incommensurable with unity, $m\alpha + 4mn$ will for each different value of n define a different direction, so that in this case there will be an infinite number of different values of $(a, \alpha)^m$, all having the same tensor a^m, and so represented by an infinite series of radii of the same circle drawn to points on the circumference, each in advance of the preceding by m × the circumference.

It will be convenient to call the vector $(a^m, m\alpha)$, corresponding to the value 0 of n, the *prime* value of the m^{th} power of the vector (a, α). Then the other values of the m^{th} power are the product of the prime value and the several values of the unit vector $(1, 4mn)$ for different values of n, or, what is the same thing, by turning the first vector through the angles $4m^{\llcorner}$, $2 \times 4m^{\llcorner}$, $3 \times 4m^{\llcorner} \ldots$ successively. If $\alpha = 0$, so that the vector becomes a positive scalar, the *prime* value of any power or root is the *arithmetical* value of the same.

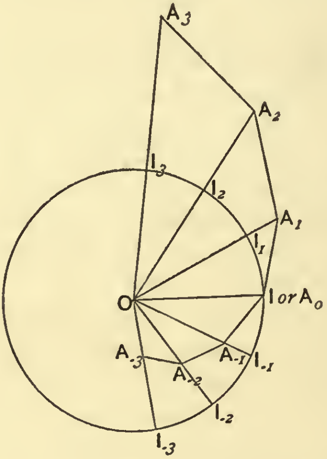
These results may be stated thus:

A scalar power of a vector has an infinite number of values, which are vectors having all the same tensor, but directions, which are all different if the index is incommensurable with unity, which are all coincident if the index is integral, and which all coincide with one or other of a set of q different directions, when the index is the fraction p/q in its lowest terms.

11. *Illustrations.*

1°. *Integral powers of the vector* (a, α).

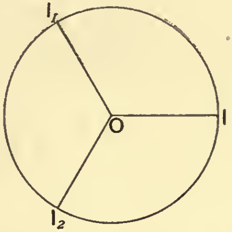
If $\overline{OA_1}$ is the vector, then the triangles A_1OA_2, A_2OA_3, . . . and $A_{-1}OI$, $A_{-2}OA_{-1}$ . . . being all similar to IOA_1, the vectors $\overline{OI}$ or $\overline{OA_0}$, $\overline{OA_1}$, $\overline{OA_2}$, $\overline{OA_3}$ . . . are the values of $\overline{OA_1}^0$, $\overline{OA_1}^1$, $\overline{OA_1}^2$, $\overline{OA_1}^3$ . . ., and the vectors $\overline{OA_{-1}}$, $\overline{OA_{-2}}$, . . . are the values of the negative powers $\overline{OA_1}^{-1}$, $\overline{OA_2}^{-2}$ . . . . It will be seen that they are radii at equal angular intervals $(\alpha\llcorner)$ of the equiangular spiral, whose pole is O, passing through the points I and A_1, and they may be said to be in vectorial geometrical progression, with the common ratio $\overline{OA_1}$ or (a, α).



If $a = 1$, the tensors are all equal to 1, the equiangular spiral becomes the unit circle, and the powers are the radii or unit vectors $\overline{OI}$, $\overline{OI_1}$, $\overline{OI_2}$ . . . and OI_{-1}, $\overline{OI_{-2}}$ . . .

2°. *Fractional powers of the vector* (1, 0) *or unity.*

Since $(1,\ 0) = (1,\ 4n)$, $(1,\ 0)^{\frac{1}{3}} = \left(1,\ \frac{4n}{3}\right)$ which has the three distinct values $(1, 0)$, $\left(1, \frac{4^{L}}{3}\right)$, $\left(1, \frac{8^{L}}{3}\right)$ for $n = 0, 1, 2$, repeated in the same order for the successive triads $(3, 4, 5)$, $(6, 7, 8)$, &c. These values are represented in the figure by the three radii of the unit circle drawn to the points I, I_1, I_2 which divide the circumference into three equal arcs. Hence unity is said to have three cube roots, of which $\overline{OI}$ or unity itself is alone scalar, the other two $\overline{OI_1}$, $\overline{OI_2}$ are unit vectors or vectors, whose tensors are unity. It should be observed that the successive integral powers of $\overline{OI_1}$ are $\overline{OI_2}$, $\overline{OI}$, $\overline{OI_1}$ repeated in the same order continually, while those of $\overline{OI_2}$ are $\overline{OI_1}$, $\overline{OI}$, $\overline{OI_2}$ repeated in the same order continually. All the integral powers of OI however are OI itself.

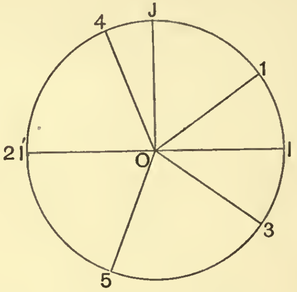


Since $(a, \alpha) = a \times (1, \alpha)$, $(a, 0)^{\frac{1}{3}} = \sqrt[3]{a} \,.\, (1, 0)^{\frac{1}{3}} = \sqrt[3]{a} \left(1, \frac{4n}{3}\right)$ denoting by $\sqrt[3]{a}$ the arithmetical cube root of the number a. Hence any positive scalar has three cube roots, which are the products of the three cube roots of 1 multiplied by $\sqrt[3]{a}$, or the vectors drawn from O to the points where OI, OI_1, OI_2 cut the circle, centre O, and radius $= \sqrt[3]{a}$.

3°. Let us examine $(1, 1^{L})^{\frac{2}{5}} = (1, 4n + 1)^{\frac{2}{5}} = \left(1, \frac{8n + 2}{5}\right)$.

In the unit circle $\overline{OJ}$, perpendicular to $\overline{OI}$, is the vector $(1, 1)$, and $\overline{OI'}$, opposite to $\overline{OI}$, is the vector $(1, 2)$ or the scalar, -1.

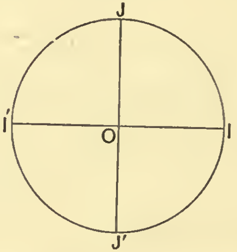
Then if, starting from OI' the circumference is divided into five equal parts by the points I', 5, 3, 1, 4, the vectors $\overline{O1}$, $\overline{O2}$ or $\overline{OI'}$,



$\overline{O3}$, $\overline{O4}$, $\overline{O5}$ are the five values of $(1,\ 1)^{\frac{2}{5}}$ or $(-1)^{\frac{1}{5}}$, corresponding to the values 0, 1, 2, 3, 4.

12. *Vector as the product of tensor and versor.*

Let the unit vector (1, 1), that is, the vector $\overline{OJ}$ whose tensor is 1 and inclination to $\overline{OI}$, reckoned in the positive sense, is a right angle, be denoted by i. Then $i^\alpha = (1,\ 1)^\alpha = (1,\ \alpha)$ and $ai^\alpha = a\,.\,(1,\ \alpha) = (a,\ \alpha)$. Hence the vector $(a,\ \alpha)$, whose tensor is a and inclination $\alpha\llcorner$ can now be expressed as the product ai^α, a



mode of expression which may henceforth naturally supersede the arbitrary one which we have hitherto adopted. Of its two factors the first a, as an operator by multiplication on another vector, alters the length or tensor of that vector in the ratio of a to

1, while the other, i^a, turns the direction of the vector through the angle a, and hence may be called (after Sir W. Hamilton) its *versor.* Thus every vector is the product of the two elements, its tensor and its versor, the former being a purely quantitative element and the latter a power of the vector i.

It should be observed that, although a is the measure of the inclination in *right angles,* the versor i^a is a factor or operator depending on the angle only and not on its particular measure with reference to any assumed unit. In fact if k denote another vector whose inclination is $m\llcorner$ so that $k = i^m$, then $i = k^{\frac{1}{m}}$ and $i^a = k^{\frac{a}{m}}$, so that the same versor may be expressed as i^a or k^β, if $\beta = \frac{a}{m}$ and β is the measure of the inclination, when the unit contains m right angles.

For example, if the unit be 1° and $k^{90} = i$, i^a and k^{90a} are different expressions for the same versor.

13. *Interpretation of* $\sqrt{-1}$.

Since $i = (1, 1)$, therefore $i^2 = (1, 2) = -1$, whence $i = \sqrt{-1}$, so that $\sqrt{-1}$, the "impossible" quantity of ordinary algebra, now appears as the unit vector OJ making with the prime vector OI a right angle in the positive sense, and in multiplication $\sqrt{-1}$ or i is the versor which turns the vector multiplied through a right angle in the positive sense.

Also $\overline{OI}'$, $\overline{OJ}'$ being the unit vectors opposite in sense to OI, OJ respectively, $i^2 = -1 = \overline{OI}'$, $i^3 = -\sqrt{-1} = \overline{OJ}'$ $i^4 = +1 = \overline{OI}$.

Hence n being any positive or negative integer,

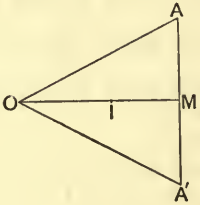
$$i^{4n} = 1 \quad i^{4n+1} = i \quad i^{4n+2} = i^2 = -1 \quad i^{4n+3} = -i.$$

14. *Uniqueness of* i.

It is also important to observe that, although the vector $\overline{OJ} = (1, 4n + 1)$ for all integral values of n, i must be regarded as denoting the unique vector $(1, 1)$ and no other of the series $(1, 5)$, $(1, 9)$ . . . which coincide with it. Otherwise i^a would in general be a function having many distinct values, whereas regarding it as the versor of a vector having a *given* value of a, it must be unique: in fact it must be the *prime value* (that is, the value when $n=0$) of $i^{(4n+1)a}$, the generalized a^{th} power of i, which determines different vectors for different values of n, except in the special case where na is an integer.

15. *Vector expressed by a complex number.*

If m, n are scalars, $m+n\sqrt{-1}$ or $m+ni$ is termed a complex number.



Let $OM=m\,.\,OI$, then $\overline{OM}=mi^0=m$, and if $\overline{MA}$ makes with the prime vector $\overline{OI}$ a right angle in the positive sense and is equal to $n.OI$, $\overline{MA}=ni$. But $\overline{OA}=\overline{OM}+\overline{MA}$, therefore $\overline{OA}=m+ni$ or the complex number $m+ni$ represents the vector $\overline{OA}$. Hence, M being the foot of the perpendicular let fall from A on the direction of the prime vector, if we call $\overline{OM}$ the *prōject* and $\overline{MA}$ the *traject* of the vector $\overline{OA}$, we have the vector expressed as the *sum* of two elements, its *prōject* and *traject*, corresponding to the so-called *possible* and *impossible* parts of the complex number. Thus the same vector may be expressed either as the *product* of its *tensor* and *versor*, or as the *sum* of its *prōject* and *traject*, and it is desirable to determine the relation between them.

Take $\quad \overline{MA'}=-\overline{MA}$, then $\overline{OA'}=m-ni$.

Also since $\quad OA'=OA$, and $MOA'=-MOA$,

if $$\overline{OA}=ai^a,\quad OA'=ai^{-a},$$

so that $\quad ai^a=m+ni$ and $ai^{-a}=m-ni$, but $ai^a ai^{-a} = a^2$ and $(m + ni)(m - ni) = m^2 - n^2 i^2 = m^2 + n^2$,

therefore $a^2 = m^2 + n^2$ or $a = \sqrt{m^2 + n^2}$,

that is, the square of the tensor = the sum of the squares of the project and traject, and the tensor is the *modulus* of the complex number, which represents the vector.

Hence also the versor $i^a = \frac{m + ni}{\sqrt{m^2 + n^2}}$, a complex number whose modulus is 1.

(Observe that we have here incidentally proved Euc. i. 47, since we have shown that $OA^2 = OM^2 + MA^2$, OMA being a right angle.)

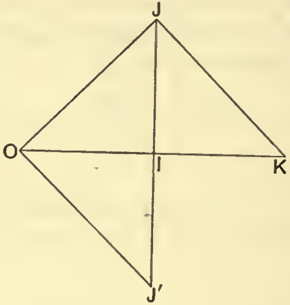
16. It has now been shown that, with the meanings of algebraical symbols and their combinations which have been found appropriate for co-planar vectors, all the fundamental laws of ordinary algebra hold good: hence all the results of ordinary algebra may be read with these new meanings, and, reserving the question of the meaning to be assigned to vector-indices for a later chapter, all such results are definitely interpretable so that the "impossibles," or "imaginaries" of ordinary algebra are now fully explained. In the succeeding chapters we shall apply our results to the establishment of Trigonometry on a purely algebraical basis. It will be instructive to conclude the present chapter with the geometrical verification of a few formulæ as illustrations of their extended meanings. It will be necessary to select only the simplest cases, because even for simple formulæ the diagrams become complicated, but the student is recommended to test others for himself by actual geometrical drawing.

17. *Illustrative examples.*

(1) $(1 + i)(1 - i) = 1 - i^2 = 1 + 1 = 2.$

In the figure $\overline{OI} = 1$, $\overline{IJ} = i$, $\overline{IJ'} = -i$,

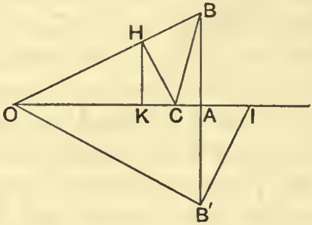
therefore $\overline{OJ} = 1 + i$, $\overline{OJ'} = 1 - i$, and if JK be drawn perpendicular to OJ to meet OI produced in K, the triangle OJK is similar to OIJ', so that $\overline{OK}=\overline{OJ}\,.\,\overline{OJ'}$,



but $\overline{OK}$ is also obviously $2\,\overline{OI}$, therefore $\overline{OJ}\,.\,\overline{OJ'}=2$, in accordance with the formula.

(2) $$(a+bi)\,(a-bi)=a^2-b^2i^2=a^2+b^2.$$

Let $$\overline{OI}=1,\ \overline{OA}=a,\ \overline{AB}=bi,\ \overline{AB'}=-bi,$$



then $$\overline{OB}=a+bi,\ \overline{OB'}=a-bi.$$

Make the angle OBC equal to OIB', then BOA being equal to $B'OA$, the triangle BOC is similar to IOB' and therefore

$$\overline{OC}=\overline{OB}\,.\,\overline{OB'}=(a+bi)\ (a-bi).$$

Draw CH perpendicular to OB and HK to OA. Then

$$OK:OA::OH:OB::OA:OI,$$

therefore $$OK\,.\,OI=OA^2 \quad \text{or} \quad OK=a^2.$$

Also $$KC : KH :: KH : OK :: AB : OA :: b : a.$$

$\therefore$ $$KC : OK :: KH^2 : OK^2 :: b^2 : a^2,$$

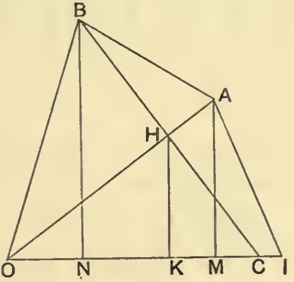
and $$OK = a^2, \quad \therefore\ KC = b^2.$$

Hence $$\overline{OC} = a^2 + b^2.$$

(3) $$(a + bi)^2 = a^2 - b^2 + 2abi.$$

Let $\overline{OM} = a$ $\overline{MA} = bi$ and therefore $\overline{OA} = a + bi$, then, if OAB be described similar to OIA, $\overline{OB} = \overline{OA}^2 = (a + bi)^2$.

Draw BHC perpendicular to OA and meeting OI in C, and draw BN HK perpendicular to OI.



Then since OH, perpendicular to BC, bisects the angle BOC, $BH = HC$ and therefore $NK = KC$ and $BN = 2HK$.

Also $KH : MA :: OH : OA :: OM : OI$ (since OIA, OAB are similar).

$\therefore$ $$KH \,.\, OI = OM \,.\, MA \quad \text{or} \quad KH = ab,$$

whence $$\overline{NB} = 2\overline{KH} = 2abi.$$

Further $$ON = OK - KN = OK - KC,$$

but $$OK : OM :: OH : OA :: OM : OI.$$

$\therefore$ $$OK \,.\, OI = OM^2 \quad \text{or} \quad OK = a^2,$$

and $$CK : KH :: KH : KO :: MA : OM :: b : a,$$

and $$KH = ab, \quad \therefore\ CK = b^2.$$

Hence $$ON = a^2 - b^2$$

so that $$\overline{OB} = \overline{ON} + \overline{NB} = a^2 - b^2 + 2abi.$$

Hence we have verified geometrically the formula

$$(a + bi)^2 = a^2 - b^2 + 2abi.$$

(4)

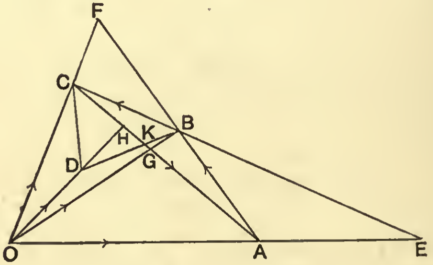
Let $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ be three vectors denoted by α, β, γ respectively,

then $$\overline{BC} = \gamma - \beta \quad \overline{CA} = \alpha - \gamma \quad \overline{AB} = \beta - \alpha,$$

and $$\overline{OA} \,.\, \overline{BC} = \gamma\alpha - \alpha\beta, \; \overline{OB} \,.\, \overline{CA} = \alpha\beta - \beta\gamma, \; \overline{OC} \,.\, \overline{AB} = \beta\gamma - \gamma\alpha$$

whence $$\overline{OA} \,.\, \overline{BC} + \overline{OB} \,.\, \overline{CA} + \overline{OC} \,.\, \overline{AB} = 0,$$

or $$\overline{OB} \,.\, \overline{AC} = \overline{OC} \,.\, \overline{AB} + \overline{OA} \,.\, \overline{BC}.$$



Hence in any quadrilateral the vector-product of the two diagonals is equal to the sum of the vector-products of the two pairs of opposite sides, due regard being paid to the senses in which the vectors are taken.

This result reduces, when O, A, B, C are concyclic, to Ptolemy's Theorem (Simson's *Euc.* vi. D).

For the inclination to the prime vector of a vector product is the sum of the inclinations of the factors, and therefore double the inclination of the bisector of the angle between them. If then the bisectors of the angles between the diagonals OB, AC and those of the angles between the two pairs of opposite sides OA, BC and OC, AB have the same inclinations or are parallel, the versors of the three vectors are equal, and the equation holds between the tensors $OB \,.\, AC = OC \,.\, AB + OA \,.\, BC$.

But these bisectors are parallel, only when O, A, B, C are concyclic, as the student may readily prove from elementary geometry, therefore Ptolemy's Theorem is proved true.

In the general case, on OC describe OCD similar to ACB,

then $$\frac{\overline{OC}}{\overline{OD}} = \frac{\overline{AC}}{\overline{AB}} \text{ or } \overline{OC} \,.\, \overline{AB} = \overline{OD} \,.\, \overline{AC},$$

and since $$\overline{OB} \,.\, \overline{AC} = \overline{OC} \,.\, \overline{AB} + \overline{OA} \,.\, \overline{BC},$$

therefore $$\overline{OB} \,.\, \overline{AC} = \overline{OD} \,.\, \overline{AC} + \overline{OA} \,.\, \overline{BC}.$$

Hence $$(\overline{OB} - \overline{OD})\, \overline{AC} = \overline{OA} \,.\, \overline{BC},$$

or $$\overline{DB} \,.\, \overline{AC} = \overline{OA} \,.\, \overline{BC},$$

from which it follows that the triangle DBC is similar to OAC. But when two vectors are equal, their tensors and versors are separately equal, so that, taking the tensors,

$$OC \,.\, AB = AC \,.\, OD \text{ and } OA \,.\, BC = AC \,.\, BD,$$

whence $OC \,.\, AB + OA \,.\, BC = AC\,(OD + BD)$ and $\therefore > AC \,.\, OB$, so that the sum of the rectangles contained by the two pairs of opposite sides is greater than the rectangle contained by the diagonals, unless O, D, B are collinear, which is the case if O, A, B, C are concyclic—a well-known extension of Ptolemy's Theorem.

Also from the equality of the versors, if OA, CB meet in E and AB, OC in F, and if OD, BD meet AC in H, K respectively, the bisectors of the angles at H, F are parallel, and likewise the bisectors of those at K, E. When O, A, B, C are concyclic, H, K merge in G the intersection of the diagonals, so that the bisectors of the angles E, F, G are parallel, as assumed above.

EXAMPLES

1. Draw the following vectors:

$$\alpha.\ \left(1,\ \frac{1}{2}\right)\quad \beta.\ (2,\ 1)\quad \gamma.\ \left(2,\ -\frac{1}{2}\right)\quad \delta.\ 3i^{\frac{1}{3}}\quad \epsilon.\ \frac{1}{3}i^{-\frac{2}{3}}\quad \zeta.\ 3+4i$$

$$\eta.\ 4+3i\quad \theta.\ \sqrt{3}-i,$$

find the projects and trajects of α, β, γ, δ, ϵ and the tensors and versors of ζ, η, θ.

2. Find the reciprocals of the vectors in the previous question, and find the products and quotients of α and β, of α and γ, of δ and ϵ, of ζ and η and of δ and θ: and verify the results geometrically.

3. If $\overline{OP}$, $\overline{Op}$ are reciprocal vectors, P, I, p, I' are concyclic, and OP, Op are equally inclined on opposite sides of OI.

4. The product $\overline{OR}$ of the vectors $\overline{OP}$, $\overline{OQ}$ may be constructed thus:

Let OP, OQ meet the unit-circle in P_0, Q_0 respectively and on OP, OQ take Oq, Op respectively equal to OQ, OP; take the arc IR_0 equal to the sum of IP_0, IQ_0: then R will be the second point of intersection of OR_0 with either of the circles R_0Qp or R_0qP.

5. If Q_0P' parallel to QP meets OP in P' and arc IR_0 = arc Q_0P_0, and if OR_0 meets the circle, centre O, and rad. OP', in R, then $OR = OP/OQ$.

6. In the figure of Euc. iv. 10, taking the base as the prime vector, express all the other vectors in the figure both by tensor and versor, and by project and traject.

7. Verify the following geometrically (α, β, being any two vectors).

$$(1)\ (-\alpha)^2=\alpha^2.\qquad (2)\ (1+i)^2=2i.\qquad (3)\ \frac{1+i}{1-i}=i.$$

$$(4)\ (1+\alpha)^2=1+2\alpha+\alpha^2.\quad (5)\ (1+\alpha)(1+\beta)=1+\alpha+\beta+\alpha\beta.$$

8. Exhibit in a figure all the values of $1^{\frac{1}{6}}, i^{\frac{1}{6}}, (-1)^{\frac{1}{6}}$, and express them as the sum of project and traject.

9. Prove by vectors that, if on $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ similar triangles AFB, BDC, AEC are similarly described, $BDEF$ is a parallelogram.

10. In the figure of Ex. 4, § 17, express the vectors $\overline{AF}$, $\overline{CE}$ in terms of α, β, γ, using (if required) the relation $l\alpha + m\beta + n\gamma = O$ as the condition for expressing any one in terms of the other two.

11. Prove that in the same figure

$$\overline{OB} \,.\, \overline{BC} \,.\, \overline{CO} + \overline{OA} \,.\, \overline{AB} \,.\, \overline{BO} = \overline{OA} \,.\, \overline{AC} \,.\, \overline{CO} + \overline{AB} \,.\, \overline{BC} \,.\, \overline{CA}.$$

12. Prove that the vector-product of the two diagonals of a parallelogram is equal to the difference of the squares of two adjacent sides. Hence shew that if on the sides AB, AD of the parallelogram $ABCD$ triangles ABE, ADF are described outside the parallelogram and similar respectively to ACB, ACD, then EF is equal and parallel to BD, and verify by geometry.

13. If the locus of a point P is a circle, and $\overline{Op}$ is the reciprocal of the vector $\overline{OP}$, the locus of p is also a circle. Prove this, and thence shew that, if $u = \dfrac{az+b}{cz+d}$, where z is the vector from O to a point which describes a circle, u is the vector of a point Q which describes another circle.

CHAPTER III

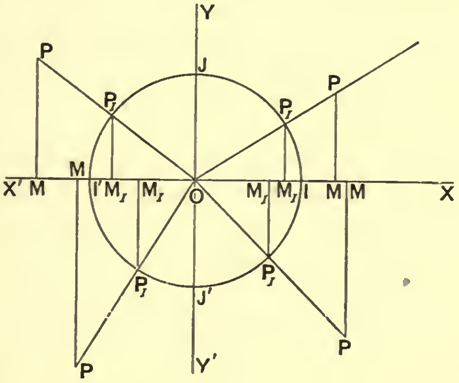
TRIGONOMETRICAL RATIOS

1. **Trigonometry**, a science deriving its name from that special branch of it in which it originated, namely, the relation between the sides and angles of a triangle, is in a more general sense the science of *periodic magnitude:* or rather of certain periodic magnitudes, that is, magnitudes whose values recur in regular sequence over and over again, as the quantity on which they depend continually increases in a given sense. This quantity is generally the angle, through which a line, revolving (in a plane) about a fixed point in it, turns in passing from a given fixed position to any other, and the magnitudes are the trigonometrical functions or ratios of the angle.

The properties of these ratios are usually deduced, and (we believe) rightly so for the beginner, from geometrical constructions; but they admit also of a complete discussion on the basis of algebra, regarded, as has been done in the preceding chapters, as the Algebra of Coplanar Vectors. This discussion will occupy the present and several succeeding chapters, in which though nothing will be assumed from the geometrical treatment (with which the reader will probably be more or less familiar) and every relation will be established *ab initio* on an algebraical basis, the more elementary relations will be succinctly treated and the geometrical developments less fully discussed than in the ordinary treatises.

2. *Definitions of the trigonometrical ratios.*

Let XOX', YOY' be two rectangular axes, that is, indefinitely extended lines at right angles to one another, and on OX let $\overline{OI}$ be taken as the prime vector or 1, and on $\overline{OY}$ let $\overline{OJ}$ be taken a unit vector (that is, one whose tensor is 1) and denoted by i.



Let $\overline{OP}$ be any other vector, whose tensor $OP = r \,.\, OI$ or r, and whose inclination is u[*], u being measured positively (counter-clockwise) or negatively (clockwise) from OI to OP as explained in the last chapter, so that $\overline{OP} = ri^u$.

From P let PM be drawn perpendicular to XOX' (as shown in the figure for four different positions of OP in the four quadrants into which XOX', YOY' divide the plane, and which are called first, second, third and fourth in the order of positive revolution, that between OX and OY being the first), and let the *project* $\overline{OM} = x \,.\, \overline{OI}$ or x and the *traject* $\overline{MP} = y \,.\, \overline{OJ}$ or yi, x, y being scalars, positive or negative according as they have the same or contrary senses to OI, OJ respectively: then, since the vector is the sum of its project and traject or $\overline{OP} = \overline{OM} + \overline{MP}$,

---

[*] In this work we shall usually denote the measure of an angle in right angles by the letters u, v, w, &c., reserving the Greek letters $\alpha, \beta \ldots \theta, \phi \ldots$ for circular measure. We shall not introduce this latter measure until we find the need for it.

$$ri^u = x + yi,$$

or
$$i^u = \frac{x}{r} + \frac{y}{r}i.$$

The positive or negative ratio $\frac{y}{r}$ is called the *sine* of the angle u, and $\frac{x}{r}$ the *cosine* of the same, or, as is usually written,

$$\frac{x}{r} = \cos u \quad \frac{y}{r} = \sin u.$$

These ratios are obviously independent of the tensor r and are functions of (that is, depend on) u only, so that without loss of generality we may take for $\overline{OP}$ the unit vector $\overline{OP_1}$ in the same direction, and then $\overline{OM_1} = \cos u$, $\overline{M_1P_1} = i \sin u$.

We have then universally the fundamental equation

$$i^u = \cos u + i \sin u,$$

so that $\cos u + i \sin u$ is the expression for the versor, whose argument is u, as a complex number. It should be observed that the versor i^u, being, as before remarked, a factor independent of the particular unit by which angles are measured, the same is true of $\cos u + i \sin u$, whence it follows that *cosine* and *sine* (and therefore all the other trigonometrical ratios) depend for their value on the angle only and not on the particular unit by which it is measured.

We have seen in the last chapter, § 14, that the tensor of $m + ni$ is $\sqrt{m^2 + n^2}$, and its versor $\frac{m + ni}{\sqrt{m^2 + n^2}}$: hence u being its inclination

$$i^u = \cos u + i \sin u = \frac{m}{\sqrt{m^2 + n^2}} + i \cdot \frac{n}{\sqrt{m^2 + n^2}},$$

whence
$$\cos u = \frac{m}{\sqrt{m^2 + n^2}}, \ \sin u = \frac{n}{\sqrt{m^2 + n^2}}.$$

From the fundamental ratios sin u, cos u, four others are derived.

$\frac{y}{x}$ or $\frac{\sin u}{\cos u}$ is called the *tangent* of u and written, tan u.

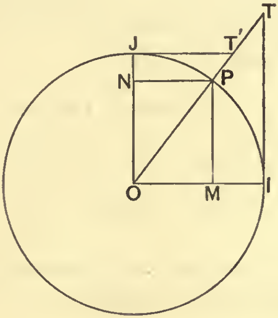
The reciprocals of the cosine, sine, and tangent are thus named :—

$\frac{r}{x}$ or $\frac{1}{\cos u}$ is called the *secant* of u and written, sec u.

$\frac{r}{y}$ or $\frac{1}{\sin u}$ is called the *cosecant* of u and written, cosec u.

$\frac{x}{y}$ or $\frac{1}{\tan u}$ is called the *cotangent* of u and written, cot u.

These ratios may be represented by lines related to the unit circle, as in the figure, where IT, JT' are tangents to the circle, thus:



MP is the sine, IT the tangent, and OT the secant, of the angle IOP or the arc IP proportional to it: POJ (or the arc PJ) is termed the *complement* of IOP (or the arc IP), then NP ($=OM$) being the sine, JT' the tangent, OT' the secant of the complement, they are termed respectively the *co*sine, *co*tangent, *co*secant of the original angle IOP (or the arc IP).

These are the original *line-definitions*, which explain the origin of the terms with the exception of the word "*sine*," which "occurs in Regiomontanus and was derived from the Arabs." *

To these are added MI, the versed sine and NJ, the co-versed sine, not now much used. It is plain that versin $u = 1 - \cos u$ and co-versin $u = 1 - \sin u$.

3. *Fundamental relations between the six ratios.*

We have then the following five equations connecting the six trigonometrical functions, by means of which any one being given the remainder may be determined:

$$i^u = \cos u + i \sin u$$

$$\sin u \operatorname{cosec} u = 1 \qquad \cos u \sec u = 1 \qquad \tan u \cot u = 1$$

$$\tan u = \frac{\sin u}{\cos u}.$$

The first of these equations requires modification to a scalar form. Since the change of sign of u reverses the sense of the traject without altering the project,

$$i^{-u} = \cos u - i \sin u,$$

whence $$(\cos u + i \sin u)(\cos u - i \sin u) = i^u \cdot i^{-u} = 1,$$

or $$\cos^2 u + \sin^2 u = 1.$$

4. The following results, easily derived from the preceding, should be regarded as known formulæ:

$$\cot u = \frac{\cos u}{\sin u} \qquad 1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \operatorname{cosec}^2 u,$$

$$\sin u = \frac{\tan u}{\sqrt{1 + \tan^2 u}} \qquad \cos u = \frac{1}{\sqrt{1 + \tan^2 u}}.$$

---

* Ball, *Short History of Mathematics*, p. 215.

5. *Expressions for the trigonometrical ratios in terms of versors.*

Since $\cos u + i \sin u = i^u$ and $\cos u - i \sin u = i^{-u}$,

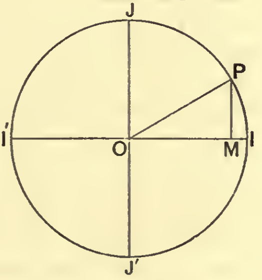
$$2 \cos u = i^u + i^{-u} \quad \text{and} \quad 2i \sin u = i^u - i^{-u},$$

or
$$\cos u = \frac{i^u + i^{-u}}{2}, \quad \sin u = \frac{i^u - i^{-u}}{2i},$$

whence
$$\tan u = \frac{1}{i}\,\frac{i^u - i^{-u}}{i^u + i^{-u}} = i\,.\,\frac{1 - i^{2u}}{1 + i^{2u}}.$$

6. *Laws of variation of the trigonometrical ratios, and their graphs.*

Suppose a unit vector $\overline{OP}$ to revolve about O from OI through OJ, OI', OJ', to OI again, then $\overline{OM}$, the cosine of IOP and $\overline{MP}$, its sine, complete a series of changes in magnitude and sense corresponding to a single period, which in subsequent revolutions are repeated over and over again in the same order; and in revolutions in the opposite sense are repeated in reversed order. If we indicate algebraical increase, or change by increments in the positive sense, by an arrowhead directed to the right (→), and the reverse by an arrowhead to the left (←), it will readily be seen that the limiting values of the several ratios for the angles 0, $1^{\llcorner}$, $2^{\llcorner}$, $3^{\llcorner}$ and their signs, and the sense of their changes between these, are those expressed in the following table:—



Angle.	0	→	$1^{\llcorner}$	→	$2^{\llcorner}$	→	$3^{\llcorner}$	→	$4^{\llcorner}$	
MP or sine	0	$\overset{+}{\rightarrow}$	1	$\overset{+}{\leftarrow}$	0	$\overset{-}{\leftarrow}$	−1	$\overset{-}{\rightarrow}$	0	Changing backwards and forwards within the limits −1 and +1.
OM or cosine	1	$\overset{+}{\leftarrow}$	0	$\overset{-}{\leftarrow}$	−1	$\overset{-}{\rightarrow}$	0	$\overset{+}{\rightarrow}$	1	
$\frac{MP}{OM}$ or tangent	0	$\overset{+}{\rightarrow}$	∞	$\overset{-}{\rightarrow}$	0	$\overset{+}{\rightarrow}$	∞	$\overset{-}{\rightarrow}$	0	May have any value from −∞ to +∞. Discontinuous, when the value is ∞; changing suddenly from −∞ to +∞ or from +∞ to −∞. The continuous changes are always in the positive sense for the tangent, in the negative for the cotangent.
$\frac{OM}{MP}$ or cotangent	∞	$\overset{+}{\leftarrow}$	0	$\overset{-}{\leftarrow}$	∞	$\overset{+}{\leftarrow}$	0	$\overset{-}{\leftarrow}$	∞	
$\frac{OM}{OP}$ or secant	1	$\overset{+}{\rightarrow}$	∞	$\overset{-}{\rightarrow}$	−1	$\overset{-}{\leftarrow}$	∞	$\overset{+}{\rightarrow}$	1	No values between −1 and +1. Discontinuous and changing sign, when the value is ∞.
$\frac{MP}{OP}$ or cosecant	∞	$\overset{+}{\leftarrow}$	1	$\overset{+}{\rightarrow}$	∞	$\overset{-}{\rightarrow}$	1	$\overset{-}{\leftarrow}$	∞	

The same results may be more clearly exhibited by means of curves or graphs, in which the abscissa ON, taken along $X'OX$ or the axis of abscissæ, is proportional to the angle, and the ordinate NQ perpendicular to the same is proportional to the ratio to be represented and the graph is the locus of Q. The graphs for one complete period from 0 to 4^L are shown in the diagrams.

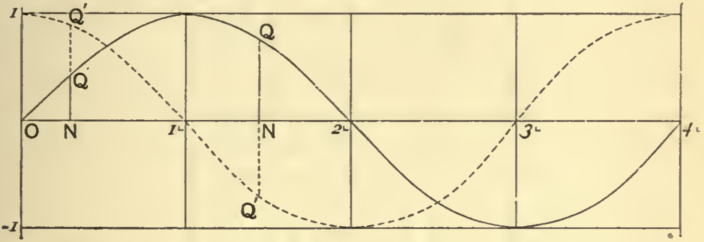


FIG. 1.

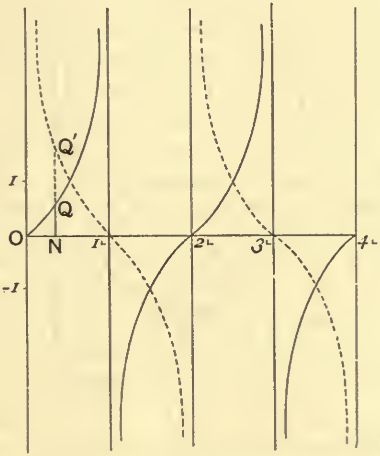


FIG. 2.

The graphs for *sine* and *cosine* are the continuous and dotted curves respectively in fig. 1, in which $ON = u$, $NQ = \sin u$ $NQ' = \cos u$.

Those for *tangent* and *cotangent* are the continuous and dotted curves respectively in fig. 2, in which $ON = u$, $NQ = \tan u$, $NQ' = \cot u$.

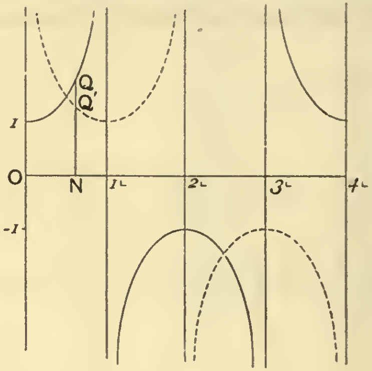


Fig. 3.

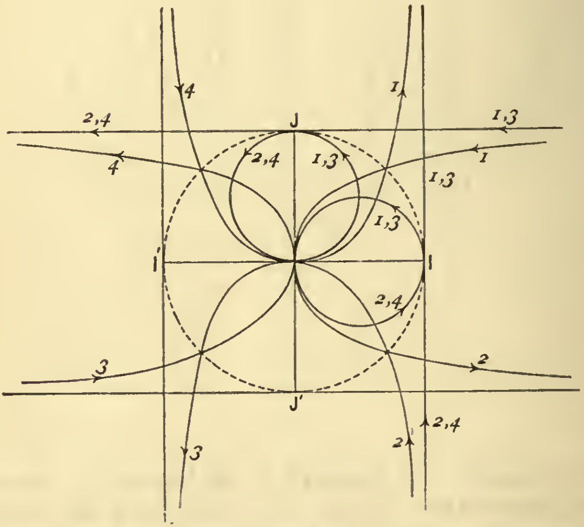


Fig. 4.

Those for *secant* and *cosecant* are the continuous and dotted lines respectively in fig. 3, in which $ON = u$, $NQ = \sec u$, $NQ' = \operatorname{cosec} u$.

Another mode of exhibiting the results is by means of the polar graphs, that is, by laying off lengths on the revolving vector $\overline{OP}$ equal to the values of the several ratios in each particular position of OP, and finding the loci of the points thus determined, the lengths being laid off from O in the sense of PO produced backwards when the values are negative (fig. 4).

Then the graphs for the cosine and sine respectively are the circles on OI, OJ as diameters, these being traversed *twice* for one complete period. The graphs for the secant and cosecant are the tangents to the unit-circle at I and J respectively, also traversed twice for a complete period. The graph for the tangent is a curve of four branches, all touching IOI' at O, to which the tangents to the unit-circle at I, I' are asymptotes: that for the cotangent a like curve touching JOJ' at O and having the tangent at J, J' for its asymptotes. (These curves are of the fourth order in rectangular coordinates.) The arrowheads indicate the sense in which the describing point is moving along the curve, as the vector revolves in the positive sense, and the numbers the quadrant to which the branch of the curve corresponds.

7. *Ratios for the reversed angle, supplement, complement, &c.*

α. The addition of any multiple (positive or negative) of four right angles leaves the ratios unchanged, since the position of vector, project, and traject are thereby unchanged, so that the ratios are *periodic* functions of the angle, the *period*, or interval of recurrence of the same values, being an angle of four right angles.

Algebraically $\quad i^{4n} = 1, \quad$ and $\therefore\ i^{4n+u} = i^u$,

whence $\quad \cos(4n+u) + i\sin(4n+u) = \cos u + i\sin u$,

so that

$$\cos(4n+u) = \cos u,\ \sin(4n+u) = \sin u,\ \tan(4n+u) = \tan u.$$

β. The reversed angle or $-u$.

Since $i^{-u} = \cos(-u) + i\sin(-u)$, and also $i^{-u} = \cos u - i\sin u$, equating projects and trajects separately,

$$\cos(-u)=\cos u,\quad \sin(-u)=-\sin u,$$

and therefore $$\tan(-u)=-\tan u.$$

γ. The angles $2+u$, and $2-u$ or the *supplement* of u. Since $i^2=-1$,

$$i^{2+u}=-\cos u-i\sin u,\quad i^{2-u}=-\cos u+i\sin u,$$

that is

$$\cos(2+u)+i\sin(2+u)=-\cos u-i\sin u,$$

$$\cos(2-u)+i\sin(2-u)=-\cos u+i\sin u\,;$$

whence

$$\begin{aligned}
\cos(2+u)&=-\cos u & \qquad \cos(2-u)&=-\cos u\\
\sin(2+u)&=-\sin u & \qquad \sin(2-u)&=\sin u.\\
\tan(2+u)&=\tan u & \qquad \tan(2-u)&=-\tan u.
\end{aligned}$$

δ. The angles $1+u$, and $1-u$ or the *complement* of u.

Since $$i^{1+u}=i.i^{u},\quad i^{1-u}=i.i^{-u},$$

$$\begin{aligned}
\cos(1+u)+i\sin(1+u)&=i(\cos u+i\sin u)\\
&=-\sin u+i\cos u,\\
\cos(1-u)+i\sin(1-u)&=i(\cos u-i\sin u)\\
&=\sin u+i\cos u\,;
\end{aligned}$$

whence

$$\begin{aligned}
\cos(1+u)&=-\sin u & \qquad \cos(1-u)&=\sin u\\
\sin(1+u)&=\cos u & \qquad \sin(1-u)&=\cos u\\
\tan(1+u)&=-\cot u & \qquad \tan(1-u)&=\cot u.
\end{aligned}$$

ϵ. The foregoing formulæ enable us to find the values of the trigonometrical ratios of any angle in terms of those of an angle in the first quadrant or an acute angle.

Thus since $1050^\circ=\frac{1050^{\llcorner}}{90}=11\frac{2^{\llcorner}}{3}=\left(12-\frac{1}{3}\right)^{\llcorner}$, the ratios for 1050° are the same as for $-\frac{1^{\llcorner}}{3}$ or -30°, so that, by β,

$$\sin(1050^\circ) = -\sin 30^\circ, \quad \cos(1050^\circ) = \cos 30^\circ,$$
$$\tan(1050^\circ) = -\tan 30^\circ.$$

So also $$-555^\circ = -\frac{555^{\llcorner}}{90} = -6\frac{1^{\llcorner}}{6} = \left(-8+2-\frac{1}{6}\right)^{\llcorner},$$

therefore, by α and γ, $\sin(-555^\circ) = \sin\frac{1^{\llcorner}}{6} = \sin 15^\circ$,

$$\cos(-555^\circ) = -\cos\frac{1^{\llcorner}}{6} = -\cos 15^\circ.$$

8. *Values of the trigonometrical ratios for certain angles.*

There are a few angles, for which the values of the sine, cosine, &c., can be easily and directly obtained from geometry. It will be instructive to see how the same can be obtained algebraically.

α. 45° or $\frac{1}{2}$.

Let $x+yi$ be a unit-vector (so that $x^2+y^2=1$), such that $(x+yi)^2=i$, or generally $(x+yi)^2=i^{4n+1}$, where n is an integer,

then $$x+yi = i^{2n+\frac{1}{2}} = \cos(2n+\tfrac{1}{2}) + i\sin(2n+\tfrac{1}{2}),$$

whence x has two values, $\cos\frac{1}{2}$ and $\cos(2+\frac{1}{2})$ or $-\cos\frac{1}{2}$, and y also two values, $\sin\frac{1}{2}$ and $\sin(2+\frac{1}{2})$ or $-\sin\frac{1}{2}$, corresponding to the two values of $2n+\frac{1}{2}$ within the first complete period. These values may be found thus:

$$(x+yi)^2 = i, \text{ and } \therefore\ x^2-y^2+2xyi = i,$$

whence $$x^2-y^2=0 \quad\text{and}\quad 2xy=1.$$

The scalar (or real) solutions of these equations are

$$x = y = \pm\frac{1}{\sqrt{2}},$$

so that $$\cos\frac{1}{2} = \sin\frac{1}{2} = \frac{1}{\sqrt{2}}.$$

β. 30° and 60° or $\frac{1^{\llcorner}}{3}$ and $\frac{2^{\llcorner}}{3}$.

Let $(x+yi)^3=1$, or generalizing $(x+yi)^3=i^{4n}$,

then $$x+yi = i^{\frac{4n}{3}} = \cos\frac{4n}{3} + i\sin\frac{4n}{3},$$

$$\therefore\ x=\cos 0 \text{ or } \cos\tfrac{4}{3} \text{ or } \cos\tfrac{8}{3} \qquad y=\sin 0 \text{ or } \sin\tfrac{4}{3} \text{ or } \sin\tfrac{8}{3}$$

$$=1 \text{ or } -\sin\tfrac{1}{3} \text{ or } -\cos\tfrac{2}{3}, \qquad =0 \text{ or } \cos\tfrac{1}{3} \text{ or } -\sin\tfrac{2}{3},$$

for the three values of $\frac{4n}{3}$ within the first period.

Now since $$(x+yi)^3=1,$$

expanding $$x^3+3x^2yi-3xy^2-y^3i=1,$$

or $$x(x^2-3y^2)+iy(3x^2-y^2)=1,$$

whence $$x(x^2-3y^2)=1 \quad\text{and}\quad y(3x^2-y^2)=0.$$

The value $y=0$ from the second equation gives $x=+1$ from the first.

Combining $3x^2-y^2=0$ with $x^2+y^2=1$,

$$x^2=\tfrac{1}{4},\quad y^2=\tfrac{3}{4},$$

whence by the first equation

$$x\left(\frac{1}{4}-\frac{9}{4}\right)=1 \quad\text{or}\quad x=-\frac{1}{2} \quad\text{and}\quad y=\pm\frac{\sqrt{3}}{2}.$$

Hence $$\sin\frac{1^{\llcorner}}{3}=\cos\frac{2^{\llcorner}}{3}=\frac{1}{2} \quad\text{and}\quad \cos\frac{1^{\llcorner}}{3}=\sin\frac{2^{\llcorner}}{3}=\frac{\sqrt{3}}{2}.$$

γ. $22\frac{1}{2}^{\circ}$ or $\frac{1}{4}^{\llcorner}$.

A similar proceeding gives, from $(x+yi)^4=i$,

$$\sin\frac{1^{\llcorner}}{4}=\cos\frac{3^{\llcorner}}{4}=\frac{1}{2}\sqrt{2-\sqrt{2}},$$

and $$\sin\frac{3^{\llcorner}}{4}=\cos\frac{1^{\llcorner}}{4}=\frac{1}{2}\sqrt{2+\sqrt{2}}.$$

δ. $\frac{1^{\llcorner}}{5}, \frac{2^{\llcorner}}{5}, \frac{3^{\llcorner}}{5}, \frac{4^{\llcorner}}{5}$, or 18°, 36°, 54°, 72°.

Let $$(x+yi)^5=1=i^{4n},\quad \therefore\ x+yi=\cos\frac{4n}{5}+i\sin\frac{4n}{5},$$

whence, taking the five values of $\frac{4n}{5}$ in the first complete period,

$$x = \cos 0 \text{ or } \cos\frac{4}{5} \text{ or } \cos\frac{8}{5} \text{ or } \cos\frac{12}{5} \text{ or } \cos\frac{16}{5}$$

$$= 1 \text{ or } \cos\frac{4}{5} \text{ or } -\cos\frac{2}{5} \text{ or } -\cos\frac{2}{5} \text{ or } \cos\frac{4}{5}$$

$$y = \sin 0 \text{ or } \sin\frac{4}{5} \text{ or } \sin\frac{8}{5} \text{ or } \sin\frac{12}{5} \text{ or } \sin\frac{16}{5}$$

$$= 0 \text{ or } \sin\frac{4}{5} \text{ or } +\sin\frac{2}{5} \text{ or } -\sin\frac{2}{5} \text{ or } -\sin\frac{4}{5},$$

so that there are *three* different (scalar) values of x and *five* of y. To determine these, we have expanding and arranging as a complex number,

$$x^5 - 10x^3y^2 + 5xy^4 + i(5x^4y - 10x^2y^3 + y^5) = 1,$$

or $\quad x(x^4 - 10x^2y^2 + 5y^4) = 1$ and $y(5x^4 - 10x^2y^2 + y^4) = 0.$

From the second equation (omitting the factor y, which gives $y=0$, $x=1$)

$$5x^4 - 10x^2y^2 + y^4 = 0,$$

which, by the condition $y^2 = 1 - x^2$, reduces to

$$16x^4 - 12x^2 + 1 = 0,$$

whence $\quad x^2 = \frac{3 \pm \sqrt{5}}{8} \quad$ and $\quad \therefore \quad y^2 = \frac{5 \mp \sqrt{5}}{8}.$

Hence it would follow that $x = \pm\frac{1 \pm \sqrt{5}}{4}$, giving four values of x, but by reason of the first of the two equations, these are reduced to two, namely $x = -\frac{1 \pm \sqrt{5}}{4}$ or $x = \frac{\sqrt{5} - 1}{4}$ or $-\frac{\sqrt{5} + 1}{4}$.

For y however there are four values, and $y = \pm\frac{\sqrt{10 \pm 2\sqrt{5}}}{4}$.

Hence $\quad \cos\frac{4^{\llcorner}}{5} = \sin\frac{1^{\llcorner}}{5} = \frac{\sqrt{5} - 1}{4}, \quad \cos\frac{2^{\llcorner}}{5} = \sin\frac{3^{\llcorner}}{5} = \frac{\sqrt{5} + 1}{4},$

$$\sin\frac{4^{\llcorner}}{5} = \cos\frac{1^{\llcorner}}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}, \quad \sin\frac{2^{\llcorner}}{5} = \cos\frac{3^{\llcorner}}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

The student may with advantage compare the foregoing with the geometrical investigations (Johnson's *Trigonometry*, pp. 60—67).

It is plain that by a similar process equations for determining the values of the ratios for other fractions of a right angle might be obtained, but the solution of these would generally depend on the solution of an equation of a degree higher than the second. The equation having definite numerical coefficients, any particular root might be obtained by Horner's process to any degree of approximation required, and thus with sufficient labour the value of the ratios for the given fraction of a right angle.

[It will be a good exercise for the student to show that $\cos \frac{1^{\llcorner}}{7} = \frac{1}{2}\sqrt{z_1}$, where z_1 is the greatest root of the equation

$$z^3 - 7z^2 + 14z - 7 = 0,$$

which, by Horner's method, will be found to be $3{\cdot}801937737 \ldots$.

so that $\cos \frac{1^{\llcorner}}{7} = \cos 12^\circ\, 51'\, 25\tfrac{5}{7}'' = {\cdot}9749279124 \ldots$.]

Ex. Find sine and cosine of $\frac{1^{\llcorner}}{6}$ and $\frac{5^{\llcorner}}{6}$.

9. *Expressions for all the angles which have the same sine, cosine, or tangent.*

Let $u_0{}^{\llcorner}$ denote a particular angle, and $u^{\llcorner}$ any other angle such that:

a. $\cos u = \cos u_0$.

Then, from the vector expressions for the cosine,

$$i^u + i^{-u} = i^{u_0} + i^{-u_0} = i^{4n+u_0} + i^{-(4n+u_0)},$$

since $i^{4n} = 1$, n being an integer. This is satisfied by

$$i^u = i^{4n+u_0} \qquad \text{or} \qquad i^u = i^{-(4n+u_0)}$$

and by these only.

Hence $u = 4n + u_0$ or $-4n - u_0$, which, since n may be either positive or negative, are both included in the formula

$$u = 4n \pm u_0.$$

β. $\sin u = \sin u_0$.

Then $$i^u - i^{-u} = i^{4n+u_0} - i^{-(4n+u_0)}$$

the two solutions of which are

$$i^u = i^{4n+u_0} \quad \text{and} \quad i^u = -i^{-4n-u_0} = i^{2-4n-u_0},$$

whence $$u = 4n + u_0 \text{ or } -(4n-2) - u_0,$$

or $u = 2n \pm u_0$, + or — according as n is even or odd, or, as it may be conveniently expressed, n being any integer

$$u = 2n + (-1)^n u_0.$$

γ. $\tan u = \tan u_0$.

Then $$\frac{i^u - i^{-u}}{i^u + i^{-u}} = \frac{i^{u_0} - i^{-u_0}}{i^{u_0} + i^{-u_0}}$$

which reduces to $i^{2u} = i^{2u_0}$, or $i^{2(u-u_0)} = 1 = i^{4n}$

therefore $$u - u_0 = 2n,$$

or $$u = 2n + u_0.$$

10. *Inverse functions.*

Let fx denote a function of x, that is, a quantity depending on x, and therefore changing when x changes, and connected with x by some definite law, which may be either algebraical, when fx denotes some ordinary algebraical expression, or transcendental, when its dependence on x cannot be expressed algebraically in finite terms (of scalars) as i^x, $\cos x$ and the other trigonometrical ratios, $\log x$ and other higher transcendents. If fx be substituted for x in fx, we have $f(fx)$ which is naturally denoted by f^2x, and then $f(f^2x)$ is denoted by f^3x, and so on.

In accordance with this notation f^0x denotes x itself, for then $f(f^0x) = fx$: and so $f^{-1}x$ denotes a quantity such that $f(f^{-1}x) = f^0x = x$, or $f^{-1}x$ is that function of x, of which the function f produces x. Any operation, which reverses a given operation on a given operand (x) so as to reproduce the operand, is termed the *inverse* of the operation (*e.g.* division is the inverse of multiplication, the square root that of the square, &c., &c.): hence $f^{-1}x$ is the function of x *inverse* to fx. It is obvious that in the same way $f^{-2}x, f^{-3}x \ldots$ are inverse to $f^2x, f^3x \ldots$ so that $f^2(f^{-2}x) = x$, $f^3(f^{-3}x) = x \ldots.$: also that $y = fx$, if $x = f^{-1}y$. The general law is (the same as that of indices) $f^m(f^nx) = f^{m+n}(x)$ for positive and negative integer values of m and n.

It is to be observed that, though the *analogy* of this notation with indices of any algebraical quantity is patent, it is only an analogy, since f by itself does not denote any quantity and has no existence independent of the subject of operation x.

[Ex. Let $fx = \dfrac{x}{1-x}$, then $f^2x = \dfrac{x}{1-2x}$, $f^3x = \dfrac{x}{1-3x}$. $f^nx = \dfrac{x}{1-nx}$

$f^{-1}x = \dfrac{x}{1+x}$, $f^{-1}x = \dfrac{x}{1+2x} \ldots f^{-n}x = \dfrac{x}{1+nx}$.]

To apply this to the trigonometrical functions. To be consistent $\cos^2 x$, $\cos^3 x \ldots.$ ought to denote $\cos(\cos x)$, $\cos(\cos(\cos x)) \ldots.$ respectively, but in practice, as these functions do not often occur, they are used, where no confusion is likely to arise, to denote the powers of $\cos x$, which are more correctly expressed as $(\cos x)^2$, $(\cos x)^3 \ldots$ Negative powers of the trigonometrical ratios are always expressed correctly as $(\cos x)^{-1}$, $(\cos x)^{-2}$, &c., &c., and the notation $\cos^{-1}x$, $\tan^{-1}x, \ldots$ is reserved to denote the functions inverse to $\cos x$, $\tan x$, &c., such that $\cos(\cos^{-1}x) = x$, $\tan(\tan^{-1}x) = x$, &c., and therefore $\cos^{-1}x$ denotes, and is read as, the angle whose cosine is x, $\tan^{-1}x$ as the angle whose tangent is x, &c. This notation is very convenient, when the angle is defined by the value of one of its trigonometrical functions instead of its measure in right angles, or degrees, and it is therefore desirable to recognize the relations established in the foregoing sections in the shape which they assume under this notation.

Thus if $\cos u = x$, $u = \cos^{-1}x$,

and since $\sin u = \sqrt{1-\cos^2 u} = \sqrt{1-x^2}$, $u = \sin^{-1}\sqrt{1-x^2}$,

also $\tan u = \dfrac{\sqrt{1-\cos^2 u}}{\cos u} = \dfrac{\sqrt{1-x^2}}{x}$, $\therefore u = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$:

also, since $$\sec u = \frac{1}{\cos u} = \frac{1}{x},\ u = \sec^{-1}\frac{1}{x},$$

and similarly $u = \operatorname{cosec}^{-1}\dfrac{1}{\sqrt{1-x^2}}$, and $u = \cot^{-1}\dfrac{x}{\sqrt{1-x^2}}$,

so that,

$$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x}$$
$$= \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}.$$

It is in like manner easily proved that

$$\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2}$$
$$= \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}.$$

The student should write down for himself the corresponding equivalents for $\sin^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$.

It is most important to observe that, even if fx is a single-valued function or one which has only one value for a given value of x, the inverse $f^{-1}x$ may be, and generally is, a multiple-valued function. Hence though $f(f^{-1}x) = x$, we can only say that some one of the values of $f^{-1}(fx) = x$ and cannot assert without limitation that $f^{-1}(fx) = f(f^{-1}x)$, or generally that, while $f^m(f^n x) = f^{m+n}(x)$, $f^n(f^m x) = f^m(f^n x)$.

Thus, as we have seen in the last article, if $\cos u = x$, u (or $\cos^{-1}x$) is one or other of the infinite number of angles included in the formula $4n \pm u_0$, where u_0 is a definite one of those angles. It will be convenient to take the angle u_0 as that one, which when defined by sin, tan, cot, or cosec, lies between -1^{L} and $+1^{\text{L}}$, or that which when defined by cos or sec lies between 0 and 2^{L}, and term it the *prime* angle of the series, and to distinguish it when written with the inverse form by making the initial letter of the function a capital. The formulæ of the last article then become,

$$\cos^{-1} x = 4n \pm \mathrm{Cos}^{-1} x \text{ or } \cos^{-1}(\cos u_0) = 4n \pm u_0$$

$$\sin^{-1} x = 2n + (-1)^n \mathrm{Sin}^{-1} x \text{ or } \sin^{-1}(\sin u_0) = 2n + (-1)^n u_0$$

$$\tan^{-1} x = 2n + \mathrm{Tan}^{-1} x \text{ or } \tan^{-1}(\tan u_0) = 2n + u_0,$$

the angle being measured in right angles and n being zero or any integer.

If an angle is defined by the value of one only of its trigonometrical ratios, even if limited to lie within an interval comprising a whole period, it will always have two distinct values, and so will generally other functions of that angle, and thus the ambiguity of sign necessarily attaching to the square roots in the formulæ above is accounted for.

Thus

$$\sin(\cos^{-1}x) = \sin(4n \pm \mathrm{Cos}^{-1} x) = \sin(\pm \mathrm{Cos}^{-1} x) = \pm\sqrt{1-x^2}$$

$$\cos(\sin^{-1}x) = \cos\left\{2n + (-1)^n \mathrm{Sin}^{-1}x\right\} = (-1)^n \cos\left\{(-1)^n \mathrm{Sin}^{-1}x\right\}$$

$$= (-1)^n \cos(\mathrm{Sin}^{-1}x) = \pm\ \overline{1-x^2}$$

$$\tan(\sin^{-1} x) = \tan\ (2n + (-1)^n \mathrm{Sin}^{-1} x) = \pm \tan\ (\mathrm{Sin}^{-1} x)$$

$$= \pm \frac{x}{\sqrt{1-x^2}}.$$

11. *Examples.*

By the formulæ in this chapter expressions involving the trigonometrical ratios of one angle may be reduced, identities connecting them verified, and trigonometrical equations involving one angle only in certain cases solved. The desired results may usually be most simply obtained by the scalar formulæ of §§ 2 and 3, of which abundant examples may be found in works on Elementary Trigonometry; but they may also be obtained by the versor expressions in § 4 and algebraical reduction, and for the sake of familiarising the student with the nature of the work involved, a few examples will be now worked out by this method.

(1) Prove that $\operatorname{cosec} u - \sin u = \cos u \cdot \cot u$,

$$\operatorname{cosec} u - \sin u = \frac{2i}{i^u - i^{-u}} - \frac{i^u - i^{-u}}{2i} = \frac{4i^2 - (i^{2u} + i^{-2u} - 2)}{2i\,(i^u - i^{-u})}$$

$$=\frac{-2-i^{2u}-i^{-2u}}{2i\,(i^{u}-i^{-u})}=-\frac{(i^{u}+i^{-u})^{2}}{2i\,(i^{u}-i^{-u})}=\frac{i^{u}+i^{-u}}{2}\,.\,\frac{i\,(i^{u}+i^{-u})}{i^{u}-i^{-u}}$$

$$=\cos u\,.\,\cot u.$$

(2) Prove that $\dfrac{\sin u}{1+\cos u}=\dfrac{1-\cos u}{\sin u}$,

$$\frac{\sin u}{1+\cos u}=\frac{i^{u}-i^{-u}}{(2+i^{u}+i^{-u})i}=\frac{\left(i^{\frac{u}{2}}-i^{-\frac{u}{2}}\right)\left(i^{\frac{u}{2}}+i^{-\frac{u}{2}}\right)}{\left(i^{\frac{u}{2}}+i^{-\frac{u}{2}}\right)^{2}.\,i}=\frac{i^{\frac{u}{2}}-i^{-\frac{u}{2}}}{i\left(i^{\frac{u}{2}}+i^{-\frac{u}{2}}\right)}$$

$$=\frac{\left(i^{\frac{u}{2}}-i^{-\frac{u}{2}}\right)^{2}}{i\,(i^{u}-i^{-u})}=\frac{i^{u}+i^{-u}-2}{i\,(i^{u}-i^{-u})}=\frac{2\cos u-2}{i\,.\,2i\sin u}=\frac{\cos u-1}{-\sin u}=\frac{1-\cos u}{\sin u}.$$

(3) $\cos^{4}u+\sin^{4}u=1-2\sin^{2}u\cos^{2}u$

$$\cos^{4}u+\sin^{4}u=\frac{(i^{u}+i^{-u})^{4}}{2^{4}}+\frac{(i^{u}-i^{-u})}{2^{4}\,i^{4}}$$

$$=\frac{1}{2^{3}}\left\{i^{4u}+i^{-4u}+6\right\}=\frac{1}{2^{3}}\left\{(i^{2u}-i^{-2u})^{2}+8\right\}$$

$$=1+\tfrac{1}{8}\,(i^{u}+i^{-u})^{2}\,.\,(i^{u}-i^{-u})^{2}=1+\tfrac{1}{8}\,(2\cos u)^{2}\,.\,(2i\sin u)^{2}$$

$$=1-2\sin^{2}u\cos^{2}u.$$

(4) Solve the equation $\tan u+\sec u=2$.

Putting the vector expressions for $\tan u$ and $\sec u$,

$$\frac{i^{u}-i^{-u}}{i\,(i^{u}+i^{-u})}+\frac{2}{i^{u}+i^{-u}}=2,$$

$$\therefore \qquad i^{u}-i^{-u}+2i=2i\,(i^{u}+i^{-u}),$$

or

$$(1-2i)\,i^{2u}+2i\,.\,i^{u}-(1+2i)=0,$$

whence
$$i^{u}=\frac{-i\pm\sqrt{i^{2}+(1-2i)(1+2i)}}{1-2i}=\frac{-i\pm\sqrt{-1+1+4}}{1-2i}$$

$$=\frac{-i\pm 2}{1-2i}=-i\,\frac{1\pm 2i}{1-2i},$$

so that $i^{u}=-i$ or $i^{u}=-i\,\dfrac{1+2i}{1-2i}=-i\dfrac{(1+2i)^{2}}{1-(-4)}=\dfrac{3i\div 4}{}$.

Therefore either $\cos u = 0$ and $\sin u = -1$, from which

$$u = (4n+3)^{\llcorner} \text{ or } n \times 360^\circ + 270^\circ,$$

or $\cos u = \frac{4}{5}$, $\sin u = \frac{3}{5}$, corresponding to a vector in the first quadrant, so that $u = 4n + \mathrm{Cos}^{-1}\frac{4}{5} = 4n + \mathrm{Sin}^{-1}\frac{3}{5}$.

Examples for Practice.

1. Verify the following, by using the vector forms for the ratios :—

(1) $\sec^2 u + \operatorname{cosec}^2 u = \sec^2 u \,.\, \operatorname{cosec}^2 u.$

(2) $(\sin u + \cos u)^2 = 1 + \sin 2u.$

(3) $\dfrac{1+\cos u}{1-\cos u} - \dfrac{1-\cos u}{1+\cos u} = 4 \cot u \operatorname{cosec} u.$

(4) $\sec 2u = \dfrac{\sec^2 u}{2 - \sec^2 u}.$

(5) $\tan^2 u = \dfrac{1-\cos 2u}{1+\cos 2u}.$

(6) $\sin u + \sin 2u = 2 \sin \dfrac{3u}{2} \cos \dfrac{u}{2}.$

(7) $\dfrac{\sin 3u - \sin u}{\cos u - \cos 3u} = \cot 2u.$

2. Solve the equations :—

(1) $\sin u + \cos u = 1.$ (2) $\cot u + \operatorname{cosec} u = 3.$

(3) $\sin 2u = \tan u.$ (4) $\cos 2u + 3 \sin u = 2.$

(5) $\tan u + \cot u = 4.$ (6) $1 + \cos u + \cos 2u + \cos 3u = 0.$

3. Prove, and verify geometrically :—

(1) $1 + i^u = 2 \cos \dfrac{u}{2} \,.\, i^{\frac{u}{2}}.$

(2) $\dfrac{1}{1 - i^u} = \dfrac{1}{2} \operatorname{cosec} \dfrac{u}{2} \,.\, i^{1-\frac{u}{2}}.$

(3) $\dfrac{1 - i^{3u}}{1 - i^u} = \dfrac{\sin \frac{3u}{2}}{\sin \frac{u}{2}} \,.\, i^u.$

(4) $\dfrac{i^u - i^v}{i^u + i^v} = i \tan \dfrac{u-v}{2}.$

CHAPTER IV

DE MOIVRE'S THEOREM, AND GENERAL TRIGONOMETRICAL FORMULÆ

1. In the last chapter from the two forms of a unit-vector or the versor of a vector, i^u and $\cos u + i \sin u$, all the relations between the trigonometrical functions of an angle have been deduced as immediate consequences, and other results, involving one angle only, obtained. The identical equality of these two forms leads at once to the theorem, known as De Moivre's Theorem, whence we shall in this chapter deduce the relations between the functions of two or more angles, angles and their multiples, and certain trigonometrical expansions.

2. *De Moivre's Theorem.*

Since $\quad i^u = \cos u + i \sin u$ and $i^v = \cos v + i \sin v$,

and by the law of indices $i^u \cdot i^v = i^{u+v}$,
therefore

$$(\cos u + i \sin u)(\cos v + i \sin v) = \cos(u+v) + i \sin(u+v).$$

Also since

$$i^{-v} = \cos v - i \sin v \quad \text{and} \quad \frac{i^u}{i^v} = i^u \cdot i^{-v} = i^{u-v},$$

therefore

$$\frac{\cos u + i \sin u}{\cos v + i \sin v} = (\cos u + i \sin u)(\cos v - i \sin v)$$
$$= \cos(u-v) + i \sin(u-v).$$

These formulæ directly express the fact that the product or quotient of two unit-vectors (or versors) is the unit-vector (or versor), whose inclination is the sum or difference respectively of their inclinations. It should be observed that, like all the formulæ of Algebra, they are universally true, whatever be the magnitude or sign of the angles u, v involved. Together with the developments which follow they constitute the famous theorem of De Moivre,* which, in the words of De Morgan (*English Cyc. Biog. Dic.*) "has had the effect of completely changing the whole character of trigonometrical science in its higher departments."

More generally, if $u_1, u_2, u_3 \ldots u_n$ denote the measure of n angles in right angles,

since $$i^{u_1} . i^{u_2} \ldots i^{u_n} = i^{u_1 + u_2 + \cdots + u_n},$$

$$(\cos u_1 + i \sin u_1)(\cos u_2 + i \sin u_2) \ldots (\cos u_n + i \sin u_n)$$
$$= \cos(u_1 + u_2 + \ldots + u_n) + i \sin(u_1 + u + \ldots + u_n).$$

Lastly, since $(i^u)^n = i^{nu}$ by the laws of indices,

$$(\cos u + i \sin u)^n = \cos nu + i \sin nu,$$

a formula which, like that from which it is derived, is quite general, and true, whatever be the value of n.

If n be integral, both sides of this equation have but one value, and the equation is unambiguous.

If $n = \frac{p}{q}$, a fraction in its lowest terms, we have seen (c. ii. § 10) that $i^{\frac{p}{q}}$, and therefore $(\cos u + i \sin u)^{\frac{p}{q}}$, has q distinct values, corresponding to q distinct directions of the unit-vector, and it is obvious that for no two of these directions can *both* project and traject be the same for each, so that we should expect that for each one of the values of u corresponding to a particular one among these directions there would be a distinct value of

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* Born at Vitry, 1667; settled in London after the revocation of the Edict of Nantes, 1685, as a teacher of mathematics; elected a Fellow of the Royal Society; published numerous papers in the *Philosophical Transactions*, besides separate works; died in London, 1754.

$\cos nu + i \sin nu$. This is easily seen to be the case; for if u_0 be a particular value of u, so that $i^u = i^{u_0}$, then $u = 4r + u_0$, where r is an integer, and therefore

$$\cos \frac{p}{q} u + i \sin \frac{p}{q} u = \cos \left(4\frac{pr}{q} + \frac{p}{q} u_0\right) + i \sin \left(4\frac{pr}{q} + \frac{p}{q} u_0\right).$$

Now the second side of this equation has q distinct values corresponding to the values $0, 1, 2 \ldots, q-1$, of r since no two of these numbers can differ by so much as q, as is necessary in order that $\frac{4pr}{q}$ and $\frac{4pr'}{q}$ may differ by a multiple of 4: and it has q *only*, since increasing r by q or any multiple of q adds a multiple of 4 to $\frac{4pr}{q}$.

Hence it is to be understood in the equation

$$(\cos u + i \sin u)^{\frac{p}{q}} = \cos \frac{p}{q} u + i \sin \frac{p}{q} u,$$

that the q values of the first side are equal to the q values of the other, each to each, the same particular value of u, out of the q values obtained by putting $r = 0, 1, 2 \ldots, \overline{q-1}$ successively in the expression $4r + u_0$, being taken on both sides.

If n be an incommensurable number, the number of different values of i^u or of different directions of the unit-vector is infinite, no two of them coinciding, and the same is true of the angles in the formula $4nr + nu_0$, since no two of these could then differ by an exact multiple of 4 right angles.

These results may be expressed thus:

The expressions $(\cos u + i \sin u)^n$ and $\cos nu + i \sin nu$ have both an infinite number of values for a given unit-vector or versor i^u, which are equal, each to each, for the same supposed value of its inclination; but these values all coincide, if n is integral; they range themselves into coincident sets of q values each, if $n = p/q$, where p is prime to q: and they are all different, if n is incommensurable.

Applications and illustrations of the above will be found in the sequel: the rest of the present chapter will be devoted to establishing other trigonometrical formulæ.

3. *Formulæ for cosine, sine, tangent of* $u \pm v$, $2u$, $2v$, *&c.*

It has been proved that, for *all* values of u, v,

$$\cos(u+v) + i\sin(u+v) = (\cos u + i\sin u)(\cos v + i\sin v),$$

whence, multiplying out,

$$\cos(u+v) + i\sin(u+v) = (\cos u \,.\, \cos v - \sin u \sin v)$$
$$+ i(\sin u \cos v + \cos u \sin v),$$

and therefore, equating projects and trajects,

$$\cos(u+v) = \cos u \cos v - \sin u \sin v,$$
$$\sin(u+v) = \sin u \cos v + \cos u \sin v.$$

Hence also

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v} = \frac{\tan u + \tan v}{1 - \tan u \tan v}.$$

In like manner from the formula

$$\cos(u-v) + i\sin(u-v) = (\cos u + i\sin u)(\cos v - i\sin v),$$

or, by changing v into $-v$ in the formulæ just obtained,

$$\cos(u-v) = \cos u \cos v + \sin u \sin v,$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v,$$
$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}.$$

The case, where $u = \frac{1}{2}\iota$ should be noted. Here, since

$$\sin u = \cos u = \frac{1}{\sqrt{2}}, \text{ and } \tan u = 1,$$

$$\cos\left(\frac{1}{2} + v\right) = \sin\left(\frac{1}{2} - v\right) = \frac{1}{\sqrt{2}}(\cos v - \sin v),$$

$$\cos\left(\frac{1}{2}-v\right)=\sin\left(\frac{1}{2}+v\right)=\frac{1}{\sqrt{2}}(\cos v+\sin v),$$

$$\tan\left(\frac{1}{2}\pm v\right)=\frac{1\pm\tan v}{1\mp\tan v}.$$

Again, since $\cos 2u+i\sin 2u=(\cos u+i\sin u)^2$,

$$\cos 2u+i\sin 2u=\cos^2 u-\sin^2 u+i\,.\,2\sin u\cos u,$$

so that $\cos 2u=\cos^2 u-\sin^2 u,\quad \sin 2u=2\sin u\cos u,$

$$\tan 2u=\frac{2\sin u\cos u}{\cos^2 u-\sin^2 u}=\frac{2\tan u}{1-\tan^2 u}.$$

The formula for $\cos 2u$ is also reducible, by the help of the relation $\cos^2 u+\sin^2 u=1$, to the following useful forms:

$$\cos 2u=2\cos^2 u-1=1-2\sin^2 u.$$

Whence also the following formulæ result, putting $\frac{u}{2}$ in the place of u:

$$\cos\frac{u}{2}=\pm\sqrt{\frac{1+\cos u}{2}},\quad \sin\frac{u}{2}=\pm\sqrt{\frac{1-\cos u}{2}},$$

and $$\tan\frac{u}{2}=\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}=\frac{2\sin\frac{u}{2}\cos\frac{u}{2}}{2\cos^2\frac{u}{2}}=\frac{\sin u}{1+\cos u};$$

or $$\tan\frac{u}{2}=\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}=\frac{2\sin^2\frac{u}{2}}{2\sin\frac{u}{2}\cos\frac{u}{2}}=\frac{1-\cos u}{\sin u},$$

so that $$\tan\frac{u}{2}=\frac{\sin u}{1+\cos u}=\frac{1-\cos u}{\sin u}=\pm\sqrt{\frac{1-\cos u}{1+\cos u}}.$$

The following formulæ for $\cos 3u$, $\sin 3u$ may be deduced by the student:

$$\cos 3u=4\cos^3 u-3\cos u,$$

$$\sin 3u=3\sin u-4\sin^3 u.$$

4. *Formulæ for converting Sums or Differences of Sines and Cosines into Products of Sines and Cosines, and conversely.*

These formulæ are readily deducible from those obtained in the last section, but it will be instructive to obtain them independently.

We have
$$\cos u + i \sin u = i^u,$$
$$\cos v + i \sin v = i^v,$$

therefore
$$(\cos u + \cos v) + i(\sin u + \sin v)$$
$$= i^u + i^v,$$
$$= i^{\frac{u+v}{2}}\left(i^{\frac{u-v}{2}} + i^{-\frac{u-v}{2}}\right),$$
$$= \left(\cos\frac{u+v}{2} + i \sin\frac{u+v}{2}\right) . 2\cos\frac{u-v}{2}.$$

Whence
$$\cos u + \cos v = 2\cos\frac{u+v}{2} . \cos\frac{u-v}{2},$$
$$\sin u + \sin v = 2\sin\frac{u+v}{2} . \cos\frac{u-v}{2}.$$

Also
$$\cos u - \cos v + i(\sin u - \sin v) = i^u - i^v = i^{\frac{u+v}{2}}\left(i^{\frac{u-v}{2}} - i^{-\frac{u-v}{2}}\right)$$
$$= \left(\cos\frac{u+v}{2} + i \sin\frac{u+v}{2}\right) . 2i\sin\frac{u-v}{2}$$
$$= -2\sin\frac{u+v}{2}\sin\frac{u-v}{2} + i . 2\cos\frac{u+v}{2}\sin\frac{u-v}{2}$$

therefore
$$\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2} = 2\sin\frac{u+v}{2}\sin\frac{v-u}{2}$$
$$\sin u - \sin v = 2\cos\frac{u+v}{2} . \sin\frac{u-v}{2}.$$

Again, since
$$2\cos u = i^u + i^{-u} \text{ and } 2i\sin u = i^u - i^{-u},$$

$$4\cos u\cos v=(i^u+i^{-u})(i^v+i^{-v})=i^{u+v}+i^{-(u+v)}+i^{u-v}+i^{-(u-v)}$$
$$=2\cos(u+v)+2\cos(u-v),$$

or
$$2\cos u\cos v=\cos(u+v)+\cos(u-v).$$

Similarly it may be proved that

$$2\sin u\sin v=\cos(u-v)-\cos(u+v)$$
$$2\sin u\cos v=\sin(u+v)+\sin(u-v)$$
$$2\cos u\sin v=\sin(u+v)-\sin(u-v).$$

5. *Sub-multiple angle formulæ.*

The formulæ just proved (with the exception of those for the functions of $u/2$) involve only single-valued functions, and are therefore without ambiguity arising from the presence of radical signs. The formulæ for the functions of sub-multiples of an angle in terms of some function of the angle, on the contrary, necessarily involve ambiguity to a degree, which it is easy to ascertain *a priori*, and thus to infer the general nature of the formula to be expected or the possibility of obtaining such a formula.

a. Functions of $u/2$ in terms of $\cos u$.

From $\cos u=\cos u_0$, we infer that $u=4n\pm u_0$ (C. iii. § 9), therefore

$$\cos\frac{u}{2}=\cos\left(2n\pm\frac{u_0}{2}\right)=(-1)^n\cos\pm\frac{u_0}{2}=\pm\cos\frac{u_0}{2}$$

$$\sin\frac{u}{2}=\sin\left(2n\pm\frac{u_0}{2}\right)=(-1)^n\sin\left(\pm\frac{u_0}{2}\right)=\pm\sin\frac{u_0}{2}$$

$$\tan\frac{u}{2}=\tan\left(2n\pm\frac{u_0}{2}\right)=\tan\left(\pm\frac{u_0}{2}\right)=\pm\tan\frac{u_0}{2}.$$

Thus in this case there is simple ambiguity of sign corresponding to the expressions found above containing a single radical sign. For a particular angle the particular sign of the radical is determined by the known sign for sine, cosine or tangent of half that angle.

β. Functions of $u/2$ in terms of $\sin u$.

From $\sin u = \sin u_0$, we infer that $u = 2n + (-1)^n u_0$, therefore

$$\sin\frac{u}{2} = \sin\left\{n + (-1)^n\frac{u_0}{2}\right\}, \quad \cos\frac{u}{2} = \cos\left\{n + (-1)^n\frac{u_0}{2}\right\},$$

and taking n successively equal to 0, 1, 2, 3, these expressions have four values, namely, two different numerical values, each with either sign, thus:

$$\sin\frac{u}{2} = \sin\frac{u_0}{2} \text{ or } \cos\frac{u_0}{2} \text{ or} -\sin\frac{u_0}{2} \text{ or} -\cos\frac{u_0}{2}$$

$$\cos\frac{u}{2} = \cos\frac{u_0}{2} \text{ or } \sin\frac{u_0}{2} \text{ or} -\cos\frac{u_0}{2} \text{ or} -\sin\frac{u_0}{2},$$

while $\tan\frac{u}{2}$ has two values only, namely, $\tan\frac{u_0}{2}$ or $\cot\frac{u_0}{2}\cdot$

To this result correspond the formulæ, which we will now investigate.

Since $\left(\cos\frac{u}{2} + \sin\frac{u}{2}\right)^2 = \cos^2\frac{u}{2} + \sin^2\frac{u}{2} + 2\sin\frac{u}{2}\cos\frac{u}{2} = 1 + \sin u,$

$$\cos\frac{u}{2} + \sin\frac{u}{2} = \pm\sqrt{1 + \sin u}.$$

Similarly $\quad \cos\frac{u}{2} - \sin\frac{u}{2} = \pm\sqrt{1 - \sin u},$

therefore $\quad 2\cos\frac{u}{2} = \pm\left\{\sqrt{1 + \sin u} \pm \sqrt{1 - \sin u}\right\}$

$$2\sin\frac{u}{2} = \pm\left\{\sqrt{1 + \sin u} \mp \sqrt{1 - \sin u}\right\},$$

each expression presenting a fourfold ambiguity corresponding to that which we were led to expect above.

The particular signs of the radicals corresponding to any particular value of the angle u may be obtained by determining the signs of $\cos u/2 + \sin u/2$ and $\cos u/2 - \sin u/2$ for such value by considering which of the two, $\cos u/2$ or $\sin u/2$, is numerically the greater, and whether it is positive or negative.

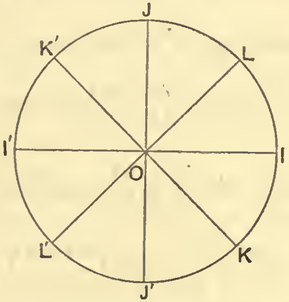
A general rule however may be obtained thus:

$$\cos\frac{u}{2}+\sin\frac{u}{2}=\sqrt{2}\sin\left(\frac{1}{2}+\frac{u}{2}\right) \text{ and } \cos\frac{u}{2}-\sin\frac{u}{2}=\sqrt{2}\cos\left(\frac{1}{2}+\frac{u}{2}\right)$$

$\therefore\ \sqrt{1+\sin u}$ must be taken with the same sign as $\sin\left(\frac{1}{2}+\frac{u}{2}\right)$

and $\sqrt{1-\sin u}$ must be taken with the same sign as $\cos\left(\frac{1}{2}+\frac{u}{2}\right)$

so that, if KOK', LOL' are the bisectors of the angles $J'OI$, IOJ respectively, the two radicals must be taken with the signs of the sine and cosine respectively corresponding to that one of the quadrants KL, LK', $K'L'$ or $L'K$ in which the terminal line of the angle $u/2$ measured from OI, lies.



[Thus if $u=\frac{7^{\llcorner}}{3}$ or 210°, so that $\sin u=\sin\left(2+\frac{1}{3}\right)=-\sin\frac{1}{3}=-\frac{1}{2}$, $\frac{u}{2}=\frac{7}{6}$ which measured from OK lies in the second quadrant,

therefore $$\cos\frac{7}{6}+\sin\frac{7}{6}=+\sqrt{1-\frac{1}{2}}=+\frac{1}{\sqrt{2}}$$

$$\cos\frac{7}{6}-\sin\frac{7}{6}=-\sqrt{1+\frac{1}{2}}=-\sqrt{\frac{3}{2}}$$

whence $$\cos\frac{7}{6}=\frac{1-\sqrt{3}}{2\sqrt{2}},\ \sin\frac{7}{6}=\frac{1+\sqrt{3}}{2\sqrt{2}}.$$]

But $\tan\frac{u}{2}=\frac{1-\cos u}{\sin u}=\frac{1\pm\sqrt{1-\sin^2 u}}{\sin u}$, a formula which gives for any particular case *two* different numerical values, both having the same sign as $\sin u$, in accordance with what we were led to expect above.

γ. As another instance, given $\sin u$ to find the functions of $u/3$.

From $\sin u=\sin u_0$, we have $u=2n+(-1)^n u_0$,

$$\text{and } \sin\frac{u}{3}=\sin\left(\frac{2n}{3}+(-1)^n\frac{u_0}{3}\right)=\sin\frac{u_0}{3} \text{ or } \sin\left(\frac{2}{3}-\frac{u_0}{3}\right)$$

$$\text{or } \sin\left(\frac{4}{3}+\frac{u_0}{3}\right) \text{ or } \sin\left(2-\frac{u_0}{3}\right) \text{ or } \sin\left(\frac{8}{3}+\frac{u_0}{3}\right) \text{ or } \sin\left(\frac{10}{3}-\frac{u_0}{3}\right),$$

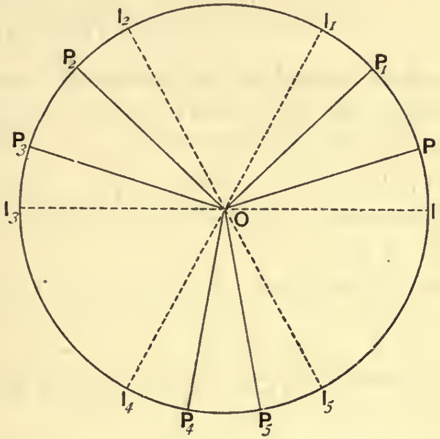
six values corresponding to the values 0, 1, 2 . . . 5 of n, which reduce to three, $\sin\frac{u_0}{3}$ or $\sin\left(\frac{2}{3}-\frac{u_0}{3}\right)$ or $-\sin\left(\frac{2}{3}+\frac{u_0}{3}\right)$, since

$$\sin\left(2-\frac{u_0}{3}\right)=\sin\frac{u_0}{3},\quad \sin\left(\frac{4}{3}+\frac{u_0}{3}\right)=\sin\left(2-\overline{\frac{2}{3}-\frac{u_0}{3}}\right)=\sin\left(\frac{2}{3}-\frac{u_0}{3}\right)$$

$$\sin\left(\frac{8}{3}+\frac{u_0}{3}\right)=-\sin\left(\frac{2}{3}+\frac{u_0}{3}\right),\quad \sin\left(\frac{10}{3}-\frac{u_0}{3}\right)=-\sin\left(\frac{2}{3}+\frac{u_0}{3}\right).$$

Hence it may be inferred that the three values of $\sin u/3$ would be obtained by the solution of a cubic equation, as is seen to be the case, since $\sin u=3\sin\frac{u}{3}-4\sin^3\frac{u}{3}$.

This result is illustrated by the diagram, in which $I, I_1, I_2, I_3, I_4, I_5$ are the extremities of unit-vectors corresponding to the angles $0, \frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}, \frac{10}{3}$ respectively, and IP being the arc corresponding to $\frac{u_0}{3}$, the arcs I_1P_1, I_2P_2, I_3P_3, I_4P_4, I_5P_5 are taken alternately backwards and forwards on the unit-circle each $=IP$, so that $\overline{OP}$, $\overline{OP_1}$ . . . $\overline{OP_5}$ are the six vectors corresponding to the inclinations contained in the formula $\frac{2n}{3}+(-1)^{n}\frac{u_0}{3}$. It is then obvious that for OP and OP_3, OP_1 and OP_2, OP_4 and OP_5, the trajects or sines



of the inclinations are equal: also that for the same three pairs, the projects or cosines are equal but of contrary signs, so that to determine $\cos u/3$ from $\sin u$, a cubic in $\cos^2 u/3$ might be expected to present itself for solution.

6. *Functions of the sum of any number of angles.*

Since

$$(\cos u_1 + i\sin u_1)(\cos u_2 + i\sin u_2)\ \ldots\ (\cos u_n + i\sin u_n)$$
$$= \cos(u_1+u_2+\ldots+u_n) + i\sin(u_1+u_2+\ldots+u_n),$$

if $S_{n-r,\,r}$ denote the sum of all the products of $n-r$ cosines and r sines of the n angles $u_1, u_2, \ldots u_n$, in the multiplication of the factors on the left-hand side of the equation the term $S_{n-r,\,r}$ will be multiplied by i^r: whence, remembering that $i^{4n}=1$, $i^{4n+2}=-1$, and $i^{4n+1}=i$, $i^{4n+3}=-i$, and grouping together scalar and non-scalar terms,

$$\cos(u_1+u_2+\ldots+u_n) + i\sin(u_1+u_2+\ldots+u_n)$$
$$= S_{n,0} - S_{n-2,2} + S_{n-4,4} - \ldots) + i(S_{n-1,1} - S_{n-3,3} + S_{n-5,5} - \ldots)$$

and therefore

$$\cos(u_1+u_2+\ldots+u_n)=S_{n,0}-S_{n-2,2}+S_{n-4,4}-S_{n-6,6}+\ldots+(-1)^r S_{n-2r,2r}+\ldots$$

$$\sin(u_1+u_2+\ldots+u_n)=S_{n-1,1}-S_{n-3,3}+S_{n-5,5}-S_{n-7,7}+\ldots+(-1)^r S_{n-2r-1,2r+1}+\ldots$$

If S_r denote the sum of all the products of r tangents of the angles, it is plain that $S_{n-r,r}=\cos u_1 \cos u_2 \ldots \cos u_n \,.\, S_r$, and the above formulæ take the form

$$\cos(u_1+u_2+\ldots+u_n)$$
$$=\cos u_1 \,.\, \cos u_2 \ldots \cos u_n \left\{1-S_2+S_4-S_6+\ldots+(-1)^r S_{2r}+\ldots\right\}$$
$$\sin(u_1+u_2+\ldots+u_n)$$
$$=\cos u_1 \,.\, \cos u_2 \ldots \cos u_n \left\{S_1-S_3+S_5-S_7+\ldots+(-1)^r S_{2r+1}+\ldots\right\}$$

and hence

$$\tan(u_1+u_2+\ldots+u_n)=\frac{S_1-S_3+S_5-\ldots+(-1)^r S_{2r+1}+\ldots}{1-S_2+S_4-\ldots+(-1)^r S_{2r}+\ldots}.$$

7. *Functions of nu in powers of cos u, sin u, tan u.*

Since $\cos nu+i\sin nu=(\cos u+i\sin u)^n$, expanding the second side by the Binomial Theorem, and grouping together [*] scalar and non-scalar terms,

$$\cos nu+i\sin nu=\cos^n u-\frac{n\,.\,\overline{n-1}}{1\,.\,2}\,.\,\cos^{n-2}u\sin^2 u$$
$$+\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,\overline{n-3}}{1\,.\,2\,.\,3\,.\,4}\cos^{n-4}u\sin^4 u-\ldots$$
$$+i\left\{n\cos^{n-1}u\sin u-\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{1\,.\,2\,.\,3}\cos^{n-3}u\sin^3 u+\ldots\right\}$$

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[*] Observe generally that if $fx=a_0+a_1x+a_2x^2+a_3x^3+\ \ldots$

$$f(xi)=a_0+a_1xi+a_2x^2i^2+a_3x^3i^3+\ \ldots$$
$$=(a_0-a_2x^2+a_4x^4-\ldots)+i(a_1x-a_3x^3+a_5x^5-\ldots),$$

since $i^{4n}=1,\ i^{4n+2}=-1,\ i^{4n+1}=i,\ i^{4n+3}=-i.$

or

$$=(\cos^n u)\left\{1-\frac{n\,.\,\overline{n-1}}{1\,.\,2}\tan^2 u+\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,\overline{n-3}}{1\,.\,2\,.\,3\,.\,4}\tan^4 u-\ldots\right.$$

$$\left.+\,i\left(n\tan u-\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{1\,.\,2\,.\,3}\tan^3 u+\ldots\right)\right\}$$

Whence, *if* $\cos^n u$ *is scalar*, as it is if n is integral,

$$\cos nu=\cos^n u\left\{1-\frac{n\,.\,\overline{n-1}}{1\,.\,2}\tan^2 u+\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,\overline{n-3}}{1\,.\,2\,.\,3\,.\,4}\tan^4 u-\ldots\right\}$$

$$=\cos^n u-\frac{n\,.\,\overline{n-1}}{1\,.\,2}\,.\,\cos^{n-2}u\,.\,\sin^2 u+\ldots$$

$$\sin nu=\cos^n u\left\{n\tan u-\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{1\,.\,2\,.\,3}\tan^3 u+\ldots\right\}$$

$$=n\,.\,\cos^n u\sin u-\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{1\,.\,2\,.\,3}\cos^{n-3}u\,.\,\sin^3 u+\ldots$$

and therefore

$$\tan nu=\frac{n\tan u-\dfrac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{1\,.\,2\,.\,3}\tan^3 u+\ldots}{1-\dfrac{n\,.\,\overline{n-1}}{1\,.\,2}\tan^2 u+\dfrac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}\tan^4 u-\ldots}$$

If n is not integral, $\cos^n u$ has a scalar value if $\cos u$ is positive, and also if $\cos u$ is negative and n is a fraction which in its lowest terms has an *odd* number for its denominator : in other cases, that is, where n is a fraction which in its lowest terms has an even number for its denominator or when n is incommensurable, $\cos^n u$ has no scalar value and the above series are not true. The following investigation gives the true result in all cases. Let the series above given for $\cos nu$ and $\sin nu$ be denoted by S, S' respectively, when $\cos u$ is positive, and by $(-1)^n S$, $(-1)^n S'$, when $\cos u$ is negative (so that S, S' are *scalar*) ; and let u_0 be the prime angle of the series for which $\cos u=\cos u_0$ and $\sin u=\sin u_0$, so that $u=4r+u_0$, r being any integer : then

$$\begin{aligned}\cos nu+i\sin nu&=\left\{\cos\overline{4r+u_0}+i\sin(4r+u_0)\right\}^n\\&=(\cos 4r+i\sin 4r)^n(\cos u_0+i\sin u_0)^n\\&=(\cos 4rn+i\sin 4rn)(S+iS')\text{ if }\cos u_0\text{ is positive.}\\&=\left\{\cos(\overline{4r+2})n+i\sin(4r+2)n\right\}(S+iS')\text{ if }\cos u_0\text{ is negative.}\end{aligned}$$

Hence

$\cos nu = S \cos 4rn - S' \sin 4rn$ or $S \cos (4r+2)n - S' \sin (4r+2)n$,
as $\cos u_0$ is positive or negative.

$\sin nu = S \sin 4rn + S' \cos 4rn$ or $S \sin (4r+2)n + S' \cos (4r+2)n$,
as $\cos u_0$ is positive or negative.

If n is integral, these expressions reduce to $\cos nu = S$, $\sin nu = S'$ if n be even; and to $\cos nu = S$ or $-S$, $\sin nu = S'$ or $-S'$, if n be odd, according as $\cos u$ is positive or negative.

If $n = \frac{p}{q}$, a fraction in its lowest terms,

$$\left.\begin{aligned} \cos nu &= S \text{ or } S \cos \frac{2p}{q} - S' \sin \frac{2p}{q} \\ \sin nu &= S' \text{ or } S \sin \frac{2p}{q} + S' \cos \frac{2p}{q} \end{aligned}\right\} \text{ when } r=0 \text{ or a multiple of } q \text{, according as } \cos u \text{ is positive or negative,}$$

and the complete expressions have q distinct values corresponding to q consecutive values of r in the series .. -3, -2, -1, 0, 1, 2, 3 . . .

[Ex. Let $u_0 = \frac{5^{\llcorner}}{3}$ or 150°, and $n = \frac{3}{4}$

so that $\cos u_0 = -\sin \frac{2}{3} = -\frac{\sqrt{3}}{2}$, $\sin u_0 = \cos \frac{2}{3} = \frac{1}{2}$, $\tan u_0 = -\frac{1}{\sqrt{3}}$,

and $nu = \frac{3}{4}\left(4r + \frac{5}{3}\right) = 3r + \frac{5}{4}$ and $(4r+2)n = (2r+1)\frac{3}{2} = 3r + \frac{3}{2}$

then
$$\cos\left(3r+\frac{5}{4}\right) = S \cos\left(3r+\frac{3}{2}\right) - S' \sin\left(3r+\frac{3}{2}\right)$$
$$\sin\left(3r+\frac{5}{4}\right) = S \sin\left(3r+\frac{3}{2}\right) + S' \cos\left(3r+\frac{3}{2}\right),$$

where
$$S = \left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}\left(1 - \frac{\frac{3}{4}\cdot-\frac{1}{4}}{1.2}\cdot\frac{1}{3} + \frac{\frac{3}{4}\cdot-\frac{1}{4}\cdot-\frac{5}{4}\cdot-\frac{9}{4}}{1.2.3.4}\cdot\frac{1}{3^2} - \dots\right)$$
$$= \sqrt[8]{\frac{27}{64}}\left(1 + \frac{3\times 1}{4.8}\cdot\frac{1}{3} - \frac{3\times 1.5.9}{4.8.12.16}\frac{1}{3^2} + \dots\right)$$
$$S' = \left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}\cdot\left(-\frac{1}{\sqrt{3}}\right)\left\{\frac{3}{4} - \frac{\frac{3}{4}\times-\frac{1}{4}\cdot-\frac{5}{4}}{1.2.3}\frac{1}{3}\right.$$
$$\left. + \frac{\frac{3}{4}\times-\frac{1}{4}\cdot-\frac{5}{4}\cdot-\frac{9}{4}\cdot-\frac{13}{4}}{1.2.3.4.5}\cdot\frac{1}{3^2} - \dots\right\}$$
$$= -\sqrt[8]{\frac{1}{192}}\left\{\frac{3}{4} - \frac{3\times 1.5}{4.8.12}\frac{1}{3} + \frac{3\times 1.5.9.13}{4.8.12.16.20}\frac{1}{3^2} - \dots\right\}$$

so that, putting r successively equal to 0, 1, 2, 3, we find

$$\cos\frac{5}{4} = -\sin\frac{1}{4} = -\frac{S+S'}{\sqrt{2}} \quad \cos\left(6+\frac{5}{4}\right) = \sin\frac{1}{4} = \frac{S+S}{\sqrt{2}}$$

$$\cos\left(3+\frac{5}{4}\right) = \cos\frac{1}{4} = \frac{S-S'}{\sqrt{2}} \quad \cos\left(9+\frac{5}{4}\right) = -\cos\frac{1}{4} = -\frac{S-S'}{\sqrt{2}}.\Big]$$

If n is incommensurable, the expressions for $\cos nu$, $\sin nu$ reduce respectively to S, S', only when $\cos u_0$ is positive and $r=0$: all the rest of the infinite number of values involve both S and S'.

It is to be observed that, unless n is a positive integer, these formulæ are applicable *only if* $\tan u$ *is between* -1 *and* $+1$, since beyond these limits the infinite series obtained from the Binomial Theorem is divergent. If $\tan u$ is numerically greater than 1, $\cot u$ is less than 1, and convergent series may similarly be obtained from the formula

$$\cos nu + i\sin nu = (\cos(4r+1)n + i\sin(4r+1)n)(\sin u_0)^n(1-i\cot u_0)^n$$

or

$$(\cos(4r-1)n + i\sin(4r-1)^n)(\sin u_0)^n(1-i\cot u_0)^n,$$

according as $\sin u$ is positive or negative. The development is left to the student.

[8. *Series for* $\cos nu$ *and* $\dfrac{\sin nu}{\sin u}$ *in descending powers of* $\cos u$.

By putting $1-\cos^2 u$ for $\sin^2 u$ in the expanded form of $(\cos u + i\sin u)^n$, it will be readily seen that we may assume

$$\begin{aligned}\cos nu + i\sin nu = {} & (A_0\cos^n u + A_2\cos^{n-2} u + A_n\cos^{n-4} u + \ldots) \\ & + i\sin u\,(A_1\cos^{n-1} u + A_3\cos^{n-3} u + \ldots)\end{aligned}$$

where A_0, A_1, $A_2\ldots$ are functions of n to be determined.

If we collect the coefficients of $\cos^n u$ from the several terms in the expanded form in § 7,

$$A_0 = 1 + \frac{n.\overline{n-1}}{1.2} + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4} + \ldots$$

but by the Binomial Theorem

$$2^n = (1+1)^n = 1 + n + \frac{n.n-1}{1.2} + \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3} + \ldots$$

and

$$0 = (1-1)^n = 1 - n + \frac{n.\overline{n-1}}{1.2} - \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3} +$$

therefore, adding $\qquad 2^n = 2A_0$ or $A_0 = 2^{n-1}$.

The remaining coefficients may be found in terms of A_0 as follows. Differentiating* both sides of the equation above with respect to u,

$$-n\sin nu + i\,n\cos nu = -(nA_0\cos^{n-1}u + \overline{n-2}\,A_2\cos^{n-3}u + \overline{n-4}\,A_4\cos^{n-5}u + \dots)\sin u$$

$$+\,i\left\{\overline{A_1\cos^n u + A_3\cos^{n-2}u + \dots} - \sin^2 u\,.\,(\overline{n-1}\,A_1\cos^{n-2}u + \overline{n-3}\,A_3\cos^{n-4}u + \dots)\right\}$$

whence, multiplying by $-i$, and putting $1-\cos^2 u$ for $\sin^2 u$,

$$n(\cos nu + i\sin nu) = \left\{nA_1\cos^n u + (\overline{n-2}\,A_3 - \overline{n-1}\,A_1)\cos^{n-2}u + (\overline{n-4}\,A_5 - \overline{n-3}\,A_3)\cos^{n-4}u + \dots\right\}$$

$$+\,i\sin u\left\{nA_0\cos^{n-1}u + \overline{n-2}\,A_2\cos^{n-3}u + \overline{n-4}\,A_4\cos^{n-5}u + \dots\right\}$$

Hence, equating the coefficients of like powers of $\cos u$ in the two expressions for $n(\cos nu + i\sin nu)$,

$$nA_1 = nA_0,\quad \overline{n-2}A_3 - \overline{n-1}\,A_1 = nA_2,\quad \overline{n-4}\,A_5 - \overline{n-3}A_3 = nA_4,\ \&c.,$$

$$nA_3 = \overline{n-2}A_2,\quad nA_5 = \overline{n-4}A_4,\quad nA_7 = (n-6)A_6,\ \&c.,$$

whence $A_1 = A_0$, $(n^2 - \overline{n-2}^2)A_3 = -\overline{n-1}\,.\,\overline{n-2}A_1$,

$$(n^2 - \overline{n-4}^2)\,A_5 = -\,\overline{n-3}\,.\,\overline{n-4}A_3, \dots$$

and generally

$$(n^2 - \overline{n-2r}^2)\,A_{2r+1} = -\,(n-2r+1)\,(n-2r)\,A_{2r-1}$$

or $$2r\,.\,2\,(n-r)\,A_{2r+1} = -\,(n-2r+1)\,(n-2r)\,A_{2r-1}.$$

We have then

$$\left.\begin{aligned}
A_1 = A_0 &= 2^{n-1}\\
2^2\,.\,1\,.\,(n-1)\,A_3 &= -\,(n-1)\,(n-2)\,A_1\\
2^2\,.\,2\,.\,(n-2)\,A_5 &= -\,(n-3)\,(n-4)\,A_3\\
2^2\,.\,3\,.\,(n-3)\,A_7 &= -\,(n-5)\,(n-6)\,A_5\\
\&c.&\\
2^2\,.\,r\,(n-r)\,A_{2r+1} &= -\,(n-2r+1)\,(n-2r)\,A_{2r-1}
\end{aligned}\right\}$$

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* In the articles in small print I have not scrupled to use the processes of the Differential Calculus when it seems the simpler and more natural course to do so. The student, who desires a proof independent of the Differential Calculus, will find an ingenious, though somewhat artificial one, in Johnson's *Trigonometry*, pp. 271–273. Serret (*Cours d'Alg. Supérieur*, Leçon 14) deduces the series from the expansion of $x^n + \frac{1}{x^n}$ in powers of $x + \frac{1}{x}$, which he investigates, and which conversely we might deduce from the series obtained above.

whence

$$A_1 = 2^{n-1}, \quad A_3 = -\frac{n-2}{1}.2^{n-3}, \quad A_5 = +\frac{\overline{n-3}.\overline{n-4}}{1.2}2^{n-5}, \text{ \&c.}$$

$$A_{2r+1} = (-1)^r \frac{(n-r-1)(n-r-2)\dots\dots(n-2r)}{1.2\dots\dots r} 2^{n-2r-1},$$

and since

$$A_{2r} = \frac{n}{n-2r} A_{2r+1},$$

$$A_{2r} = (-1)^r \frac{n.(n-r-1)(n-r-2)\dots(n-2r+1)}{1.2.3\dots r} 2^{n-2r-1},$$

so that

$$2\cos nu = (2\cos u)^n - \frac{n}{1}.(2\cos u)^{n-2} + \frac{n.\overline{n-3}}{1.2}(2\cos u)^{n-4} + \dots$$

$$+ (-1)^r \frac{n.(n-r-1)(n-r-2)\dots(n-2r+1)}{1.2.3\dots r}(2\cos u)^{n-2r} +, \text{ \&c.}$$

$$\frac{\sin nu}{\sin u} = (2\cos u)^{n-1} - \frac{n-2}{1}.(2\cos u)^{n-3}$$

$$+ \frac{\overline{n-3}.\overline{n-4}}{1.2}.(2\cos u)^{n-5} - \dots$$

$$+ (-1)^r \frac{(n-r-1)(n-r-2)\dots(n-2r)}{1.2.3\dots r}(2\cos u)^{n-2r-1} + \dots$$

These series can only be used, when n is a positive integer; for if n is not a positive integer, they become infinite series which are essentially divergent, the limiting ratio of one term to the preceding being $\sec^2 u$, which is always greater than 1.]

[9. Series for $\cos nu$, $\sin nu$ is ascending powers of $\sin u$ or $\cos u$. In the expanded form of $(\cos u + i \sin u)^n$, if $+\sqrt{1-\sin^2 u}$ be substituted for $\cos u$ (supposed positive) and its powers expanded by the Binomial Theorem, the series may be arranged according to ascending powers of $\sin u$ in such a manner that we may assume

$$\cos nu + i \sin nu = A_0 + A_2 \sin^2 u + A_4 \sin^4 u + \dots$$

$$+ i(A_1 \sin u + A_3 \sin^3 u +$$

where $A_0, A_1, A_2 \dots$ are functions of n to be determined, and it is easily seen that $A_0 = 1$, $A_1 = n$.

Differentiate twice, then since *

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* Strictly it is only true for *circular measure* that $\frac{d}{dx}(\sin x) = \cos x$, &c. For any other measure $\frac{d}{dx}(\sin x) = k \cos x$, &c., where k is a constant dependent on the unit of angular measure. In the investigation above this constant would appear on both sides of the equation, and so may be omitted.

$$\frac{d^2}{dx^2}(\cos nx) = -n^2\cos nx, \quad \frac{d^2}{dx^2}(\sin nx) = -n^2\sin nx,$$

and $$\frac{d^2}{dx^2}(\sin^r x) = \frac{d}{dx}(r\sin^{r-1}x\cos x) = r\,.\,\overline{r-1}\sin^{r-2}x\,.\cos^2 x - r\sin^r x$$

$$= r\,.\,\overline{r-1}\sin^{r-2}x - r^2\sin^r x.$$

$$-n^2(\cos nu + i\sin nu) = A_2\,.\,(2\,.\,1 - 2^2\sin^2 u) + A_4(4\,.\,3\sin^2 u - 4^2\sin^4 u)$$
$$+ A_6(6\,.\,5\sin^4 u - 6^2\,.\sin^6 u) + \dots$$
$$+ i\left\{-A_1\sin u + A_3(3\,.\,2\sin u - 3^2\sin^3 u) + \dots\right\}$$
$$= 1\,.\,2\,A_2 + (3\,.\,4\,A_4 - 2^2 A_2)\sin^2 u + (5\,.\,6\,A_6 - 4^2 A_4)\sin^4 u + \dots$$
$$+ i\left\{(2\,.\,3\,A_3 - 1^2 A_1)\sin u + (4\,.\,5\,A_5 - 3^2 A_3)\sin^3 u + \dots\right\}$$

whence, equating the coefficients of like powers of $\sin u$,

$$A_0 = 1,\ 1\,.\,2\,A_2 = -n^2 A_0,\ 3\,.\,4\,A_4 = -(n^2 - 2^2)A_2,$$
$$5\,.\,6\,A_6 = -(n^2 - 4^2)A_4, \text{ \&c.}$$
$$A_1 = n,\ 2\,.\,3\,A_3 = -(n^2 - 1^2)A_1,\ 4\,.\,5\,A_5 = -(n^2 - 3^2)A_3,$$
$$6\,.\,7\,A_7 = -(n^2 - 5^2)A_5, \text{ \&c.,}$$

and therefore

$$\cos nu + i\sin nu = 1 - \frac{n^2}{1\,.\,2}\sin^2 u + \frac{n^2\,.\,(n^2 - 2^2)}{1\,.\,2\,.\,3\,.\,4}\sin^4 u$$
$$- \frac{n^2(n^2 - 2^2)(n^2 - 4^2)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}\sin^6 u + \dots$$
$$+ i\left\{n\sin u - \frac{n\,.\,(n^2 - 1^2)}{1\,.\,2\,.\,3}\sin^3 u + \frac{n\,.\,(n^2 - 1^2)(n^2 - 2^2)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}\sin^5 u - \dots\right\}$$

Hence, if n is integral,

$$\cos nu = 1 - \frac{n^2}{1\,.\,2}\sin^2 u + \frac{n^2(n^2 - 2^2)}{1\,.\,2\,.\,3\,.\,4}\sin^4 u$$
$$- \frac{n^2(n^2 - 2^2)(n^2 - 4^2)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}\sin^6 u + \text{\&c.,}$$

a series, which is finite only if n is even, but a convergent infinite series, if n is odd: and

$$\sin nu = n\sin u - \frac{n\,.\,(n^2 - 1^2)}{1\,.\,2\,.\,3}\,.\sin^3 u + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\sin^5 u -, \text{ \&c.,}$$

a series which is finite only if n is odd, but a convergent infinite series, if n is even.

If n is fractional or incommensurable, observations similar to those in § 7 apply.

Putting $1-u$ for u in the above series, since $\sin(1-u)=\cos u$,

$$\cos(n-nu)=(-1)^{\frac{n}{2}}\cos nu,\ \sin(n-nu)=(-1)^{\frac{n}{2}-1}\sin nu, \text{ if } n \text{ is even,}$$

and $\cos(n-nu)=(-1)^{\frac{n-1}{2}}\sin nu$, $\cos(n-nu)=(-1)^{\frac{n-1}{2}}\cos nu$, if n is odd,

$$(-1)^{\frac{n}{2}}\cos nu = 1 - \frac{n^2}{1\,.\,2}\cos^2 u + \frac{n^2\,(n^2-2^2)}{1\,.\,2\,.\,3\,.\,4}\cos^4 u - \dots$$

which is a finite series, if n is even:

$$(-1)^{\frac{n-1}{2}}\cos nu = n\cos u - \frac{n\,.\,(n^2-1^2)}{1\,.\,2\,.\,3}\cos^3 u + \frac{n\,(n^2-1^2)\,(n^2-3^2)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\cos^5 u - \dots$$

a finite series, if n is odd.]

10. *Series for $\cos^n u$, $\sin^n u$ in terms of cosines or sines of multiples of u.*

These may be deduced from the versor expressions for cosine and sine, namely, $2\cos u = i^u + i^{-u}$ $2i\sin u = i^u - i^{-u}$ as follows. Let n be any positive integer, then from the Binomial Theorem,

$$2^n\cos^n u = (i^u + i^{-u})^n = i^{nu} + n i^{(n-2)u} + \frac{n\,.\,\overline{n-1}}{1\,.\,2}\,i^{(n-4)u} + \dots + n i^{-(n-2)u} + i^{-nu},$$

or taking the terms in pairs, the first with the last, the second with the last but one and so on,

$$2^n\cos^n u = i^{nu} + i^{-nu} + n\left(i^{\overline{n-2u}} + i^{-\overline{n-2u}}\right) + \frac{n\,.\,\overline{n-1}}{1\,.\,2}\left(i^{\overline{n-4u}} + i^{-(n-4)u}\right) + \dots,$$

the last term being

$$\frac{n\,.\,\overline{n-1}\,\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,\ldots\,\frac{n}{2}}\quad\text{or}\quad\frac{n\,.\,\overline{n-1}\,\ldots\,\frac{n+3}{2}}{1\,.\,2\,\ldots\,\frac{n-1}{2}}\,(i^{u}+i^{-u}),$$

according as n is even or odd.

Whence, putting $2\cos ru$ for $i^{ru}+i^{-ru}$ and reversing the order of the terms, so that they may stand in the order of their numerical importance,

$$2^n\cos^n u=\frac{n\,.\,\overline{n-1}\,\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,\ldots\,\frac{n}{2}}+\frac{n\,.\,\overline{n-1}\,\ldots\left(\frac{n}{2}+2\right)}{1\,.\,2\,\ldots\left(\frac{n}{2}-1\right)}\,.\,2\cos 2u$$

$$+\ldots+n\,.\,2\cos\overline{n-2}\,u+2\cos nu,\ \text{if } n \text{ is even,}$$

and

$$2^n\cos^n u=\frac{n\,.\,\overline{n-1}\,\ldots\,\frac{n+3}{2}}{1\,.\,2\,\ldots\,\frac{n-1}{2}}\,.\,2\cos u+\frac{n\,.\,\overline{n-1}\,\ldots\,\frac{n+5}{2}}{1\,.\,2\,\ldots\,\frac{n-3}{2}}\,.\,2\cos 3\,u$$

$$+\ldots+n\,.\,2\cos\overline{n-2}\,u+2\cos nu,\ \text{if } n \text{ is odd.}$$

It should be observed that, in the first of these series all the terms except the first, and in the second all the terms, are *periodic* and go through a complete cycle of changes, as the angle u is continuously increased by four right angles, the terms containing u, $2u$, $\ldots$ ru, having gone through $1, 2 \ldots r$ complete cycles respectively in that interval. Hence $\cos^n u$ is a function, which is purely *periodic*, if n is odd; but which, if n is even, has a non-periodic part, which is termed its *mean value*. We may state then that the mean value of $\cos^n u$ is 0 or $\dfrac{n\,.\,\overline{n-1}\,\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,\ldots\,\frac{n}{2}}\,.\,\dfrac{1}{2^n}$, according as n is odd or even. This last term may be put into the form $\frac{1.3.5\ldots(2r-1)}{2.4.6\ldots 2r}$, if $n=2r$.

In the series above, if we change u into $1-u$, we find

$$2^n\sin^n u=\frac{n.\overline{n-1}\ldots\left(\frac{n}{2}+1\right)}{1.2\ldots\frac{n}{2}}-\frac{n.\overline{n-1}\ldots\left(\frac{n}{2}+2\right)}{1.2\ldots\left(\frac{n}{2}-1\right)}2\cos 2u$$

$$+\ldots-(-1)^{\frac{n}{2}}n.2\cos\overline{n-2}\,u+(-1)^{\frac{n}{2}}.2\cos nu,\text{ if } n \text{ is even}:$$

and

$$2^n\sin nu=\frac{n.\overline{n-1}\ldots\frac{n+3}{2}}{1.2\ldots\frac{n-1}{2}}.2\sin u-\frac{n.\overline{n-1}\ldots\frac{n+5}{2}}{1.2\ldots\frac{n-3}{2}}.2\sin 3u$$

$$+\ldots-(-1)^{\frac{n-1}{2}}n.2\sin\overline{n-2u}+(-1)^{\frac{n-1}{2}}.2\sin nu,\text{ if } n \text{ is odd.}$$

Thus also $\sin^n u$ is purely periodic, if n is odd; but has a mean value if n is even, the same as that of $\cos^n u$, as might have been anticipated *a priori*.

It will be a good exercise for the student to deduce the series for $\sin^n u$ independently from the expression $2i\sin u=i^u-i^{-u}$ by a course of reasoning similar to that employed for $\cos^n u$ above.

11. *Expression of $\cos^m u.\sin^n u$ in terms of sines or cosines of multiples of u.*

It is frequently required, in the expansions employed in astronomical and other similar investigations, to reduce a term containing the factor $\cos^m u\sin^n u$ into a series of sines or cosines of multiples of u of the same character as those found for $\cos^n u$, $\sin^n u$ in the last article. No general formula of a simple character can be given for this purpose, but an easy process applicable to any particular case may be readily obtained as follows. Denoting the coefficients in the expansion of $(1+x)^m$ by the Binomial Theorem by 1_m, 2_m, $3_m\ldots$, we have

$$2^m \cos^m u = (i^u + i^{-u})^m = i^{mu} + 1_m \,.\, i^{(m-2)u} + 2_m \,.\, i^{(m-4)u} + \dots + 1_m \,.\, i^{-(m-2)u} + i^{-mu}$$

and

$$2^n i^n \sin^n u = (i^u - i^{-u})^n = i^{nu} - 1_n\, i^{(n-2)u} + 2_n\, i^{(n-4)u} - \dots \mp 1_m\, i^{-(n-2)u} \pm i^{-nu},$$

the upper sign corresponding to n even, the lower to n odd.

Hence it is plain that the product $2^{m+n} i^n \cos^m u \sin^n u$ will be equal to a series of the general form

$$i^{(m+n)u} + A_2\, i^{(m+n-2)u} + A_4\, i^{(m+n-4)u} + \dots \pm A_4\, i^{-(m+n-4)u} \mp A_2\, i^{(m+n-2)u} \pm i^{-(m+n)u},$$

reducing, if n is even, to a series of *cosines* of multiples of u commencing with $(m+n)u$ and diminishing by $2u$ from term to term: but, if n is odd, reducing to a like series of *sines*. Further, if $m+n$ is even, there will be a single middle term of the form $A_{m+n}i^0$ or A_{m+n}; but if $m+n$ is odd, there will be two middle terms grouping into one of the form $A_{m+n-1}\,(i^u \pm i^{-u})$. Hence if m, n are both even or both odd, the series would appear to consist of a *mean* term and *periodic* terms of cosines or sines respectively of $2u$, $4u \dots (m+n)u$. It will be seen however that, when m and n are both odd, the middle term A_{m+n}, being the sum of pairs of equal terms with contrary signs, will vanish, so that the mean term is 0 or the series is periodic. If one is even and the other odd, the series will be purely periodic and involve cosines or sines of u, $3u$, $5u \dots (m+n)u$, according as n is even or odd.

To obtain the coefficients of the several terms, we observe that

$$\begin{aligned} 2^{m+n}\, i^n \cos^m u \cos^n u &= (i^u + i^{-u})^m \,.\, (i^u - i^{-u})^n \\ &= (i^{2u} - i^{-2u})^m (i^u - i^{-u})^{n-m}, \text{ if } m < n \\ &= (i^{2u} - i^{-2u})^n\, (i^u + i^{-u})^{m-n}, \text{ if } m > n: \end{aligned}$$

then the result may be obtained by expanding $(i^{2u} - i^{2u})^m$ and multiplying the expansion $(n-m)$ times by $i^u - i^{-u}$, if $m < n$, or by expanding $(i^{2u} - i^{-2u})^n$ and multiplying $(m-n)$ times by $i^u + i^{-u}$, if $m > n$.

In the expansion by the Binomial Theorem of $(i^{2u} - i^{-2u})^m$. the indices in the successive terms being written

$$2mu,\ (2m-2)u,\ (2m-4)u\ \dots\ -(2m-4)u,\ -(2m-2)u,\ -2mu,$$

the corresponding coefficients are

$$1,\ 0,\ -m\ \dots\ -m,\ 0,\ 1\ :$$

and to form the coefficients of the product of such a series by $i^u + i^{-u}$ or $i^u - i^{-u}$, it is only necessary to add each coefficient to the next following without or with change of sign of the first of the two.

The process is best shown by an example, for which let us take $\cos^8 u \sin^4 u$.

Then

$$2^{12}\, i^4 \cos^8 u \,.\, \sin^4 u = (i^u + i^{-u})^8 (i^u - i^{-u})^4 = (i^u + i^{-u})^4 \,.\, (i^{2u} - i^{-2u})_4.$$

The coefficients in the expansion of $(i^{2u} - i^{-2u})^4$ of the terms involving

$$i^{8u}\quad i^{6u}\quad i^{4u}\quad i^{2u}\quad i^{0}\quad i^{-2u}\quad i^{-4u}\quad i^{-6u}\quad i^{-8u},$$

are respectively $\quad 1\ \ 0\ -4\ \ 0\ \ 6\ \ 0\ -4\ \ 0\ \ 1,$

and multiplying four times in succession by $i^u + i^{-u}$, we have the following table of coefficients :

$$1 + 0 - 4 + 0 + 6 + 0 - 4 + 0 + 1$$

$$1 + 1 - 4 - 4 + 6 + 6 - 4 - 4 + 1 + 1$$

$$1 + 2 - 3 - 8 + 2 + 12 + 2 - 8 - 3 + 2 + 1$$

$$1 + 3 - 1 - 11 - 6 + 14 + 14 - 6 - 11 - 1 + 3 + 1$$

$$1 + 4 + 2 - 12 - 17 + 8 + 28 + 8 - 17 - 12 + 2 + 4 + 1,$$

whence

$$\begin{aligned}2^{12}\, i^4 \cos^8 u \sin^4 u = {} & (i^{12u} + i^{-12u}) + 4\,(i^{10u} + i^{-10u}) + 2\,(i^{8u} + i^{-8u}) \\ & - 12\,(i^{6u} + i^{-6u}) - 17\,(i^{4u} + i^{-4u}) + 8\,(i^{2u} + i^{-2u}) + 28,\end{aligned}$$

$$\therefore\ 2^{11}\cos^8 u\sin^4 u = 14 + 8\cos 2u - 17\cos 4u - 12\cos 6u$$
$$+ 2\cos 8u + 4\cos 10u + \cos 12u,$$

and hence the mean value of $\cos^8 u \sin^4 u = \dfrac{14}{2^{11}} = \dfrac{7}{1024}$.

From the same table we may infer that

$$2^7\cos^4 u\sin^4 u = 3 - 4\cos 4u + \cos 8u$$
$$2^8\cos^5 u\sin^4 u = 6\cos u - 4\cos 3u - 4\cos 5\,u + \cos 7u + \cos 9u$$
$$2^9\cos^6 u\sin^4 u = 6 + 2\cos 2u - 8\cos 4u - 3\cos 6u$$
$$+ 2\cos 8u + \cos 10u$$
$$2^{10}\cos^7 u\sin^4 u = 14\cos u - 6\cos 3u - 11\cos 5u - \cos 7u$$
$$+ 3\cos 9u + \cos 11u.$$

The student should verify the following.

For $\cos^3 u \sin^7 u$ the table is

$$1 + 0 - 3 + 0 + 3 + 0 - 1$$
$$1 - 1 - 3 + 3 + 3 - 3 - 1 + 1$$
$$1 - 2 - 2 + 6 + 0 - 6 + 2 + 2 - 1$$
$$1 - 3 + 0 + 8 - 6 - 6 + 8 + 0 - 3 + 1$$
$$1 - 4 + 3 + 8 - 14 + 0 + 14 - 8 - 3 + 4 - 1,$$

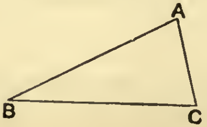
whence

$$2^6\cos^3 u\sin^4 u = 3\cos u - 3\cos 3u - \cos 5u + \cos 7u$$
$$2^7\cos^3 u\sin^5 u = 6\sin 2u - 2\sin 4u - 2\sin 6u + \sin 8u$$
$$2^8\cos^3 u\sin^6 u = 6\cos u - 8\cos 3u + 3\cos 7u - \cos 9u$$
$$2^9\cos^3 u\sin^7 u = 14\sin 2u - 8\sin 4u - 3\sin 6u + 4\sin 8u - \sin 10u.$$

12. *Formulæ connecting the sides and angles of a triangle.*

In the triangle ABC we have, by vector addition,

$$\overline{BC} = \overline{BA} + \overline{AC}.$$



Let a, b, c be the tensors of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively or the lengths of the sides opposite to the angles A, B, C of the triangle respectively, and let the prime vector coincide in direction with $\overline{BC}$, then the inclination of $\overline{BA}$ is the angle B and that of $\overline{AC}$ is $-C$: hence

$$a = ci^{B} + bi^{-C}$$

or

$$\begin{aligned} a &= c(\cos B + i \sin B) + b(\cos C - i \sin C) \\ &= (c \cos B + b \cos C) + i(c \sin B - b \sin C) \end{aligned}$$

whence, equating projects and trajects,

$$a = c \cos B + b \cos C \text{ and } c \sin B - b \sin C = 0,$$

also

$$\begin{aligned} a^2 &= (c \cos B + b \cos C)^2 + (c \sin B - b \sin C)^2 \\ &= c^2(\cos^2 b + \sin^2 B) + b^2(\cos^2 C + \sin^2 C) + 2bc(\cos B \cos C \\ &\qquad - \sin B \sin C) \\ &= b^2 + c^2 + 2bc \cos(B + C), \end{aligned}$$

or

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

since $B + C$ is the supplement of A.

These formulæ form the basis of the applications of Trigonometry to the Solution of Triangles, the determination of Heights and Distances, the relations of a triangle to its associated points and circles, and other geometrical problems, which do not fall within the scope of the present Treatise. For these the student is referred to the ordinary treatises on Elementary Trigonometry, and for fuller development, especially with reference to the more recent geometry of the triangle, to those of Casey and Johnson.

EXAMPLES.

1. Prove the following identities:

α. $\cos 3u = 4 \cos u \cos (u + \tfrac{2}{3}) \cos (u - \tfrac{2}{3})$.

β. $\tan 3u = -\tan u \tan (u + \tfrac{2}{3}) \tan (u - \tfrac{2}{3})$.

γ. $\sin 5u = 16 \sin u \sin (u - \tfrac{2}{5}) . \sin (u + \tfrac{2}{5}) \cos (u - \tfrac{2}{5}) \cos (u + \tfrac{2}{5})$.

δ. $\tan 5u = \tan u . \tan (u - \tfrac{2}{5}) \tan (u + \tfrac{2}{5}) . \cot (u - \tfrac{2}{5}) \cot (u + \tfrac{2}{5})$.

2. Prove that

α. $$\begin{aligned}\cos 6u &= 1 - 18 \sin^2 u + 48 \sin^4 u - 32 \sin^6 u \\ &= -1 + 18 \cos^2 u - 48 \cos^4 u + 32 \cos^6 u.\end{aligned}$$

β. $$\begin{aligned}\sin 6u &= \cos u (6 \sin u - 32 \sin^3 u + 32 \sin^5 u) \\ &= \sin u (6 \cos u - 32 \cos^3 u + 32 \cos^5 u).\end{aligned}$$

γ. The general term in the expansion of $\sin 6u$ in powers of $\sin u$ is

$$-6 . \frac{1 . 3 . 5 \dots . (2r-7) \times (2r+3)(2r+5)}{2 . 4 . 6 \dots . 2r} \sin^{2r+1} u.$$

3. If $\cos v = \dfrac{\cos u - e}{1 - e \cos u}$, $\tan \dfrac{v}{2} = \pm \sqrt{\dfrac{1+e}{1-e}} \tan \dfrac{u}{2}$. Explain the ambiguous sign.

4. If $u + v + w = 2$,

$$\cot u + \cot v + \cot w = \cot u \cot v \cot w + \operatorname{cosec} u \operatorname{cosec} v \operatorname{cosec} w$$
$$\text{and } \cot v \cot w + \cot w \cot u + \cot u \cot v = 1.$$

5. If $\sin^3 z = \sin (u - z) . \sin (v - z) . \sin (w - z)$ and $u + v + w = 2$, then $\cot z = \cot u + \cot v + \cot w$ and $\operatorname{cosec}^2 z = \operatorname{cosec}^2 u + \operatorname{cosec}^2 v + \operatorname{cosec}^2 w$.

6. $16 \cos^3 u \cos^2 2u = \cos 7u + 3 \cos 5u + 5 \cos 3u + 7 \cos u$.

7. $16 \sin^2 u \sin^3 2u = \sin 8u - 2 \sin 6u - 2 \sin 4u + 6 \sin 2u$.

In the following examples (Nos. 8 to 11) let x, a, b, c be the unit vectors i^{2z}, i^{2u}, i^{2v}, i^{2w} respectively: then

8. $\dfrac{x - a}{b - a} = i^{z-v} \dfrac{\sin (z - u)}{\sin (v - u)}$.

9. From the identity $\dfrac{x - a}{b - a} + \dfrac{x - b}{a - b} = 1$, prove that

$$\begin{aligned}\sin (z - u) \cos (z - v) - \sin (z - v) \cos (z - u) &= \sin (v - u), \\ \sin (z - u) \cos (z + v) - \sin (z - v) \cos (z + u) &= \sin (v - u) \cos 2z, \\ \sin (z - u) \sin (z + v) - \sin (z - v) \sin (z + u) &= \sin (v - u) \sin 2z.\end{aligned}$$

10. If $S(b - c)$ denote the sum of all terms involving a, b, c of the type $b - c$, so that $S(b - c) = 0$ and $S\{a(b - c)\} = 0$, prove that

$$S\{\cos (v + w) \sin (v - w)\} = 0,\quad S\{\cos u \sin (v - w)\} = 0,$$
$$S\{\sin (v + w) \sin (v - w)\} = 0,\quad S\{\sin u \sin (v - w)\} = 0.$$

11. From $(b-c)(c-a)(a-b)=S\{a^2(c-b)\}=S\{a(b^2-c^2)\}=S\{bc(c-b)\}$, deduce the formulae

$$S\{\sin 2(v-w)\}=-4\sin(v-w)\sin(w-u)\sin(u-v),$$

$$S\{\cos 3u \overset{\sin}{\underset{\cos}{}}(v-w)\}=4\sin(v-w)\sin(w-u)\sin(u-v)\overset{\sin}{\underset{\cos}{}}(u+v+w),$$

$$S\{\sin(v-w)\cos(v+w-2u)\}=4\sin(v-w)\sin(w-u)\sin(u-v),$$

$$S\{\sin(v-w)\sin(v+w-2u)\}=0.$$

12. If $u+v+w=2s$,

$$\cos s\,.\cos\overline{s-u}\,.\cos\overline{s-v}\,.\cos\overline{s-w}+\sin s\,.\sin\overline{s-u}\,.\sin\overline{s-v}\,.\sin\overline{s-w}$$
$$=\cos u\,.\cos v\,.\cos w.$$

13. Prove that

$$S\{\cos^3 u\sin^3(v-w)\}=3\cos u\cos v\cos w\sin(v-w)\sin(w-u)\sin(u-v)$$
$$S\{\sin^3 u\sin^3(v-w)\}=3\sin u\sin v\sin w\sin(v-w)\sin(w-u)\sin(u-v).$$

14. If $u=\mathrm{Tan}^{-1}\frac{b}{a}$, $\frac{a+ib}{b+ia}=\sin 2u-i\cos 2u$,

and $\left(\frac{a-ib}{b-ia}\right)^n-\left(\frac{a+ib}{b+ia}\right)^n=(-1)^{\frac{n}{2}-1}2i\sin 2nu$ or $(-1)^{\frac{n-1}{2}}2i\cos(2nu)$, as n is even or odd.

Verify these results by a diagram.

15. Prove that $m+ni=\sqrt{m^2+n^2}\,.\,i^{\mathrm{Tan}^{-1}\frac{n}{m}}$ or $\sqrt{m^2+n^2}\,.\,i^{2+\mathrm{Tan}^{-1}\frac{n}{m}}$, according as m is positive or negative.

16. The expression $(4+3i)^{\frac{1}{2}}+(4-3i)^{\frac{1}{2}}$ has the four values

$$\pm 2\sqrt{5}\cos(\tfrac{1}{2}\mathrm{Tan}^{-1}\tfrac{3}{4}),\ \pm 2\sqrt{5}\sin\{\tfrac{1}{2}\mathrm{Tan}^{-1}\tfrac{3}{4}\}\,.\,i.$$

Prove this, and verify by a diagram.

17. The expression $(m+ni)^{\frac{1}{3}}+(m-ni)^{\frac{1}{3}}$ has 9 values, which are (if $\sqrt{m^2+n^2}=\rho$ and $\mathrm{Tan}^{-1}\frac{n}{m}=3u$) $2\rho\cos u$, $2\rho\cos u\,.\,i^{\frac{4}{3}}$, $2\rho\cos u\,.\,i^{-\frac{4}{3}}$,

$$2\rho\cos(u-\tfrac{2}{3})\,.\,i^{\frac{2}{3}},\quad -2\rho\cos(u-\tfrac{2}{3}),\ 2\rho\cos(u-\tfrac{2}{3})\,.\,i^{-\frac{2}{3}},$$
$$2\rho\cos(u+\tfrac{2}{3})i^{\frac{2}{3}},\ 2\rho\cos(u+\tfrac{2}{3})\,.\,i^{-\frac{2}{3}},\quad -2\rho\cos(u+\tfrac{2}{3}).$$

Verify this by a diagram.

18. The expression $(m+ni)^{\frac{p}{q}}+(m-ni)^{\frac{p}{q}}$, where p is prime to q, has q^2 different values included in the form

$$2(m^2+n^2)^{\frac{p}{2q}}\cos\left\{\frac{p}{q}\mathrm{Tan}^{-1}\frac{n}{m}+\frac{2\lambda p}{q}\right\}i^{\frac{2\lambda' p}{q}},$$

where λ, λ′ are independent integers: and that q of these values are scalar.

CHAPTER V

VECTOR INDICES AND LOGARITHMS

1. In the previous chapters the form A^B has been considered, subject to the restriction that the index B represents a positive or negative scalar. It remains to examine the general case, where the index is unrestricted so that B, as well as A, represents a vector or complex number.

The primary definition of an index for a positive integer, having been found to be meaningless when the index was supposed to be a positive or negative scalar not integral, has been extended in ordinary algebra to the latter case by finding the *interpretation* which it is necessary to give to an index in that case, in order that the laws of indices deduced from the primary definition may still hold good. The definition thus extended again fails, when the index is supposed to be a complex number or vector, and we must have recourse to a similar process of interpretation to obtain a complete definition.

We have then to investigate the meaning of A^B, where A and B are both vectors or complex numbers, subject to the conditions that the laws of Indices, as well as the general laws of Algebraical Combinations, still hold good. These laws are the three expressed in the forms following :

$$(1)\ a^m . a^n = a^{m+n}, \quad (2)\ (a^m)^n = a^{mn}, \quad (3)\ (ab)^m = a^m b^m,$$

to which we may add that $a^0 = 1$, and the definition of a logarithm that, if $a^m = b$, $m = \log_a b$.

2. *k^i, where k is a positive scalar.*

Beginning with the case of the vector power i of any positive scalar k, let us assume that (if possible)* it is equal to some vector ri^u, so that $k^i = ri^u$.

In this equation r and u, if one of them is not an absolute constant, must be functions of k only. Let us assume farther that there is a value of k which will reduce the vector ri^u to i: then, denoting this value by η,

$$\eta^i = i.$$

Hence, since $k = \eta^{\log_\eta k}$, by Index Law (2) above,

$$k^i = (\eta^{\log_\eta k})^i = (\eta^i)^{\log_\eta k} = i^{\log_\eta k},$$

so that $$r = 1 \text{ and } u = \log_\eta k,$$

or k^i is a unit vector, whose inclination to the prime vector (expressed in right angles) is $\log_\eta k$.

We have therefore

$$k^i = \eta^{i \log_\eta k} = i^{\log_\eta k} = \cos(\log_\eta k) + i \sin(\log_\eta k),$$

in which it remains to determine the numerical value of η.

[Or, we may proceed thus:

Let $$k^i = \phi k \,.\, i^{\psi k},$$

then, since $$(k^n)^i = (k^i)^n,$$

$$\phi(k^n) \,.\, i^{\psi(k^n)} = (\phi k)^n \,.\, i^{n\psi k};$$

so that $$\phi(k^n) = (\phi k)^n \text{ and } \psi(k^n) = n\psi k.$$

Now, since $$\log(k^n) = n \log k \text{ and } \psi(k^n) = n\psi k,$$

$$\frac{\psi(k^n)}{\log(k^n)} = \frac{\psi k}{\log k},$$

so that, since, n having any value, k^n has any value, the value of $\frac{\psi k}{\log k}$ is a constant independent of k; and this may obviously be taken as 1 by properly taking the base of the system of logarithms, hitherto left arbitrary: hence, denoting the base by η,

---

* As far as *a priori* considerations are concerned, this assumption might have turned out to be impossible, in which case k^i would have been, relatively to our subject-matter, coplanar vectors, uninterpretable or an "impossible" quantity.

$$\psi k = \log_\eta k.$$

Also, since $$\phi(k^n) = (\phi k)^n,$$

$$\log\left\{\phi(k^n)\right\} = n \log \phi(k),$$

whence, by similar reasoning ($\log\phi$ taking the place of ψ)

$$\frac{\log(\phi k)}{\log k} = \text{const.} = m \text{ suppose},$$

and therefore $\phi k = k^m$, where m is arbitrary.

Hence $k^i = k^m i^{\log_\eta k}$, where m is an arbitrary number.

If $\qquad k = \eta,\ \eta^i = \eta^m i = i,$ if $\eta^m = 1$ or $m = 0$.]

3. Since $\quad \cos u = \dfrac{i^u + i^{-u}}{2}$ and $\sin u = \dfrac{i^u - i^{-u}}{2i}$,

we have, by putting η^i for i, the new and important formulæ

$$\cos u = \frac{\eta^{iu} + \eta^{-iu}}{2} \quad \text{and} \quad \sin u = \frac{\eta^{iu} - \eta^{-iu}}{2i}.$$

From the latter of these

$$\frac{\sin u}{u} = \frac{\eta^{iu} - \eta^{-iu}}{2iu},$$

both sides of which take the form $^0/_0$, if we suppose u to vanish: but, as we proceed to show, each has a finite limiting value in this case, and by the equality of these limits the value of η will be determined.

4. *Limit of* $\dfrac{\sin u}{u}$ *when u vanishes.*

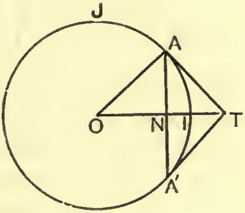
If $\overline{OI}$ be the prime vector and $\overline{OA}$ the unit vector i^u, the angle IOA, subtending the arc IA on the unit circle, contains u right angles, so that, angles at the centre of a circle being in the same ratio as the arcs on which they stand,

$$u = \frac{\text{angle } IOA}{\text{a right angle}} = \frac{\text{arc } IA}{\text{quadrant } IJ} = \frac{\text{arc } IA}{\frac{1}{2}\pi},$$

where π denotes the ratio of the circumference of a circle to its diameter, a numerical constant whose value (as we shall show hereafter) is 3·1415926. . . Also, if AN is drawn perpendicular to OI,

so that $$\sin u = NA,\ \frac{\sin u}{u} = \frac{\pi}{2}\frac{NA}{\text{arc } IA},$$

and $\lim\limits_{u=0}\left(\frac{\sin u}{u}\right) = \frac{\pi}{2}.\ \text{limit}\ \frac{NA}{\text{arc } IA}$, when the arc IA vanishes.



Now if AN produced meets the unit circle in A' and tangents are drawn at A and A', they will meet OI produced in the same point T, and the arc AIA' and its chord ANA' are bisected in I and N respectively, and AT, $A'T$ are equal. Hence, assuming that an arc of a circle is greater than its chord and less than the sum of the tangents at its extremities,

$$NA < \text{arc } IA < TA,$$

and therefore $\frac{NA}{TA}$, $\frac{\text{arc } IA}{TA}$, and 1, are ratios in ascending order of magnitude.

But, from the similar triangles ATN, OAN,

$$\frac{NA}{TA} = \frac{ON}{OA} = \frac{ON}{OI},$$

and ultimately, when the arc IA vanishes, N coincides with I, therefore the limit of $\frac{NA}{TA}$, and also that of $\frac{\text{arc } IA}{TA}$, is 1.

Hence
$$\text{limit}_{u=0} \frac{\sin u}{u} = \frac{\pi}{2}$$

5. *Circular measure of an angle. Radian.*

The value of the limit just found for $\sin u/u$ depends on the unit of measurement having been taken as a *right angle.* It will often be convenient in the sequel to adopt such a measure for an angle as will make the limit 1 instead of $\pi/2$, and this will be the case if the measure of the angle IOA is taken as $\frac{\pi}{2} u$ instead of u. But $\frac{\pi}{2} u$ measures the length of the arc IA on the unit circle, so that on this system the measure of an angle at the centre of the unit circle, is the length of the arc on which it stands, and consequently the unit angle is that which stands on the unit arc. This measure of an angle is termed *circular measure,* and the unit is termed a *radian,* for which we shall adopt the symbol $^{\odot}$. If instead of a unit circle a circle of any radius is taken, since in different circles arcs subtended by equal angles are in the ratio of the radii, the circular measure of an angle at the centre is equal to the ratio $\frac{\textit{arc subtended}}{\textit{radius}}$, and the radian is that angle which subtends an arc equal to the radius.

It follows from the above that a right angle $= \pi/2$ radians; and to compare the measures of an angle in right angles, degrees, and radians respectively, we have

$$2^{\llcorner} = 180^{\circ} = \pi^{\odot},$$

so that if a given angle contains u right angles, or A degrees, or a radians, or
$$u^{\llcorner} = A^{\circ} = a^{\odot},$$
$$\frac{u}{2} = \frac{A}{180} = \frac{a}{\pi};$$

by means of which either of these measures may be converted into another.

We have also $\text{limit}_{a=0}\left(\frac{\sin a}{a}\right) = 1.$

6. *Limit*$_{z=0}\dfrac{a^z-1}{z}=\log_\epsilon a$.

Another limit of fundamental importance is that of $\dfrac{a^z-1}{z}$ when z vanishes. It is plain that, if finite, it must be a function of a, so that we may write

$$\text{limit}_{z=0}\frac{a^z-1}{z}=\phi a,$$

where the form of ϕ has to be determined.

Putting a^n in the place of a, we have

$$\phi\,(a^n)=\text{limit}_{z=0}\frac{a^{nz}-1}{z}=n\times\text{limit}_{nz=0}\frac{a^{nz}-1}{nz}=n\phi a,$$

and we know that $\qquad \log\,(a^n)=n\log a.$

$$\therefore\frac{\phi\,(a^n)}{\log\,(a^n)}=\frac{\phi a}{\log a};$$

whence, since n and therefore a^n may have any value whatever, $\phi a/\log a$ is a constant independent of a, so that, C denoting this constant, $\qquad \phi a=C\log a,$

The base of the logarithm above has been left indefinite, and the value of C will obviously depend on the base chosen.

Let e denote that base for which $C=1$, then

$$\text{limit}_{z=0}\frac{a^z-1}{z}=\phi a=\log_\epsilon a,$$

where e is a numerical constant to be determined from the condition that $\qquad \text{limit}_{z=0}\dfrac{e^z-1}{z}=1.$

7. *Determination of e.*

Let ϵ be a quantity such that $\dfrac{\epsilon^z-1}{z}=1$ and therefore

$$\epsilon=(1+z)^{\frac{1}{z}},$$

then $e = \text{limit}_{z=0}\,\epsilon = \text{limit}_{z=0}(1+z)^{\frac{1}{z}} = \text{limit}_{m=\infty}\left(1+\frac{1}{m}\right)^m$,

if $z = \frac{1}{m}$, where m may be taken to be a positive integer.

Expanding by the Binomial Theorem and reducing,

$$\left(1+\frac{1}{m}\right)^m = 1+1+\frac{1\,.\left(1-\frac{1}{m}\right)}{1\,.\,2}+\frac{1\,.\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1\,.\,2\,.\,3}+\ldots$$

$$+\frac{1\,.\left(1-\frac{1}{m}\right)\ldots\left(1-\frac{r-1}{m}\right)}{1\,.\,2\ldots r}+\ldots$$

Now $a, b, c\,..$ being positive quantities all less than 1,

$$(1-a)(1-b)(1-c)\ldots > 1-(a+b+c+\ldots)$$

For $\quad (1-a)(1-b) = 1-a-b+ab > 1-\overline{a+b}$

$\therefore \quad (1-a)(1-b)(1-c) > (1-\overline{a+b})(1-c) > 1-\overline{a+b+c}$

and so on.

Hence $\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\ldots\left(1-\frac{r-1}{m}\right) > 1-\frac{1+2+\ldots+\overline{r-1}}{m}$

$$> 1-\frac{r\,.\,(r-1)}{2m}$$

and this product is positive and <1, since r is not greater than m, so that, θ_r being some positive fraction less than 1, we may write

$$\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\ldots\left(1-\frac{r-1}{m}\right) = 1-\theta_r\frac{r\,.\,\overline{r-1}}{2m}.$$

Therefore

$$\left(1+\frac{1}{m}\right)^m = 1+1+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+\ldots+\frac{1}{1\,.\,2\ldots m}$$

$$-\frac{1}{2m}-\frac{\theta_3}{1}\cdot\frac{1}{2m}-\frac{\theta_4}{1\,.\,2}\cdot\frac{1}{2m}\ldots-\frac{\theta_m}{1\,.\,2\ldots\overline{m-2}}\,\frac{1}{2m},$$

so that $\left(1+\frac{1}{m}\right)^m$ differs from the sum of the $\overline{m+1}$ terms

$$1+1+\frac{1}{1\,.\,2}+\ldots+\frac{1}{1\,.\,2\ldots m}$$

by less than

$$\frac{1}{2m}\left(1+1+\frac{1}{1\,.\,2}+\ldots+\frac{1}{1\,.\,2\ldots\overline{m-2}}\right), \text{ since } \theta_3, \theta_4, \ldots \theta_m \text{ are } < 1.$$

Now the series above has its terms less than those of the series

$$1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{m-1}},$$

of which the limit, when m is increased without limit, is

$$1+\frac{1}{1-\frac{1}{2}} \text{ or } 3:$$

therefore this series has a finite limit less than 3.

Hence

$$\left(1+\frac{1}{m}\right)^m \text{ differs from } 1+1+\frac{1}{1\,.\,2}+\ldots+\frac{1}{1\,.\,2\ldots m}$$

$$\text{by less than } \frac{3}{2m},$$

and when m is increased without limit, this difference vanishes, so that

$$e=\operatorname{limit}_{m=\infty}\left(1+\frac{1}{m}\right)^m=1+1+\frac{1}{1\,.\,2}+\ldots+\frac{1}{\underline{|r}}+\ldots \text{ (ad inf.)}$$

and e is thus proved to be a number greater than 2 and less than 3.

The series for e converges with tolerable rapidity, and it is easy to calculate by taking a sufficient number of terms that to 9 fractional places

$$e=2{\cdot}718281828\ldots$$

a result which the student should verify.

It is easy to prove that e is an incommensurable number. For if not, let $e = \frac{m}{n}$ where m and n are integers, then

$$e\,.\,\lfloor\underline{n} = m\,.\,\lfloor\underline{n-1} = \text{integral terms} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots$$

so that we have two integers differing by

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots$$

which is less than

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots \text{ or } \frac{1}{(n+1)\left(1 - \frac{1}{n+1}\right)} \text{ or } \frac{1}{n}:$$

that is, the difference of two integers is a fraction, which is absurd.

8. *Value of* $\eta = e^{\frac{\pi}{2}}$

Resuming the equation (§ 3),

$$\frac{\sin u}{u} = \frac{\eta^{iu} - \eta^{-iu}}{2iu}:$$

since
$$\frac{\eta^{iu} - \eta^{-iu}}{2iu} = \eta^{-iu}\frac{\eta^{2iu} - 1}{2iu}$$

and, when $u = 0$, limit $\eta^{-iu} = 1$, and by § 6 limit $\frac{\eta^{2iu} - 1}{2iu} = \log_e \eta$,

while by § 4,
$$\text{limit}\,\frac{\sin u}{u} = \frac{\pi}{2},$$

we have
$$\frac{\pi}{2} = \log_e \eta \text{ or } \eta = e^{\frac{\pi}{2}},$$

so that
$$\log_{10}\eta = \frac{\pi}{2}\log_{10}e.$$

Taking $\pi = 3{\cdot}1415926\ldots$ and $\log_{10}e = {\cdot}43429448\ldots$

$$\log_{10}\eta = {\cdot}6821881\ldots$$

and
$$\eta = 4{\cdot}810475\ldots$$

Hence the remarkable result that

$$(4{\cdot}810475\ldots)^i = i.$$

9. *Exponential Expressions for the Trigonometrical Functions.*

To sum up the results of the preceding investigations, if an angle contain u right angles or θ radians, so that $\theta = \frac{\pi}{2}u$, we have

$$\cos u + i \sin u = i^u = \eta^{iu} = e^{i\theta}$$

$$\cos u = \cos\theta = \frac{i^u + i^{-u}}{2} = \frac{\eta^{iu} + \eta^{-iu}}{2} = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin u = \sin\theta = \frac{i^u - i^{-u}}{2i} = \frac{\eta^{iu} - \eta^{-iu}}{2i} = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

and $\therefore$
$$\tan u = \tan\theta = \frac{1}{i}\,\frac{i^{2u} - 1}{i^{2u} + 1} = \frac{1}{i}\cdot\frac{\eta^{2iu} - 1}{\eta^{2iu} + 1} = \frac{1}{i}\,\frac{e^{2i\theta} - 1}{e^{2i\theta} + 1}.$$

10. *Vector Power of a Vector.*

We are now in a position to interpret generally the expression A^B, where A and B are both vectors.

Let r, u be the tensor and versor respectively of the vector A,

so that
$$A = ri^u,$$

and let m, n be the project and the traject of the vector B,

so that
$$B = m + ni,$$

then, since
$$\eta^i = i,$$

$$A^B = (ri^u)^{m+ni} = (r\eta^{iu})^{m+ni} = r^m\, r^{in}\, \eta^{imu}\, \eta^{i^2nu} = r^m\, \eta^{-nu}\, \eta^{i(n\log_\eta r + mu)}$$

or
$$A^B = r^m \eta^{-nu}\, i^{n \log_\eta r + mu}\,;$$

or, with complete generality, putting $u + 4\lambda$ for u (λ an integer),

$$A^B = r^m \eta^{-nu} \eta^{-4\lambda n} i^{n \log_\eta r + mu}\, i^{4\lambda m}.$$

Hence A^B is a vector, whose tensor is $r^m \eta^{-nu} \eta^{-4\lambda u}$, which for different values of λ positive and negative gives the terms of an infinite geometric series continued upwards and downwards from the fundamental term $r^m \eta^{-nu}$ with the common ratio η^{-4n}: and whose versor angle, expressed in right angles, is

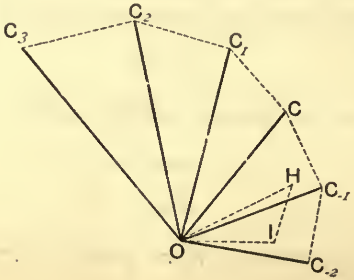
$$n \log_\eta r + mu + 4\lambda m,$$

which in like manner gives the terms of an infinite arithmetic series continued upwards and downwards from the fundamental term $n \log_\eta r + nu$ with the common difference $4m$.

The same may be more concisely expressed thus:

The values of A^B form infinite series of vectors in vectorial geometric progression formed from the fundamental vector $r^m \eta^{-nu} i^{n \log_\eta r + mu}$ forwards and backwards by the common vectorial ratio $\eta^{-4n} i^{4m}$.

This leads to the following geometrical interpretation.



Taking $\overline{OI}$ as the prime vector, and $\overline{OC}$ as the fundamental vector, so that the tensor $OC = r^m \eta^{-nu}$ and the versor angle $IOC = (n \log_\eta r + mu)^\iota$: also taking $\overline{OH}$ as the vectorial common ratio, so that the tensor $OH = \eta^{-4n}$ and the versor angle $IOH = 4m^\iota$; on OC describe the triangle OCC_1, similar to OIH, OC being homologous to OI and the angle COC_1, measured in the same sense as IOH, and on OC_1, similarly a second triangle OC_1C_2, and the like on OC_2 and so on; and farther on OC describe the triangle OCC_{-1} similar to OHI, OC being homologous to OH the angle COC_{-1} measured in the sense opposite to that of IOH, and on OC_{-1} similarly a second triangle $OC_{-1}C_{-2}$ and the like on OC_{-2} and so on: then the infinite series of vectors

$$\ldots \overline{OC}_{-3}\ \overline{OC}_{-2}\ \overline{OC}_{-1}\ \overline{OC}\ \overline{OC}_1\ \overline{OC}_2\ \overline{OC}_3 \ldots,$$

in which
$$\overline{OC}_\lambda = \overline{OC} \,.\, \overline{OH}^\lambda,$$

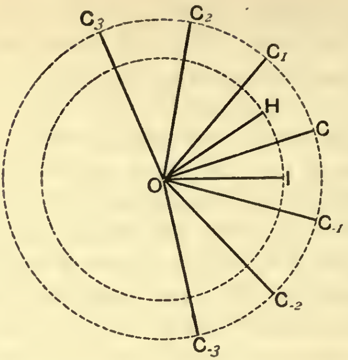
constitute the several values of A^B.

It is easily seen that these vectors are radii of the equiangular spiral, whose equation is $r = OC \,.\, \eta^{-\frac{n}{m}u}$ for the values, $u = 4\lambda m$.

11. *Particular Cases.*

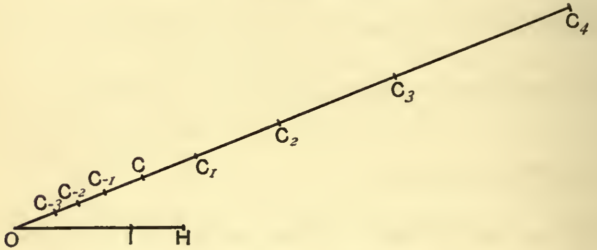
(1) If $n = 0$, the index is scalar and the case has been discussed in Chap. ii. Regarded as a particular case of the general interpretation above, the vectors $OC_1, OC_2, \ldots$ have all the same tensor r^m; which are coincident if m is an integer, but, if $m = p/q$ (p being prime to q) form an infinite series of periods of q distinct vectors, each inclined to the next at an angle of $\frac{4}{q}$ right angles; while, if m is incommensurable, they form an infinite series without periodic repetition. In the diagram annexed

$$\overline{OC} = r^m i^{mu}, \overline{OH} = i^{4m}.$$

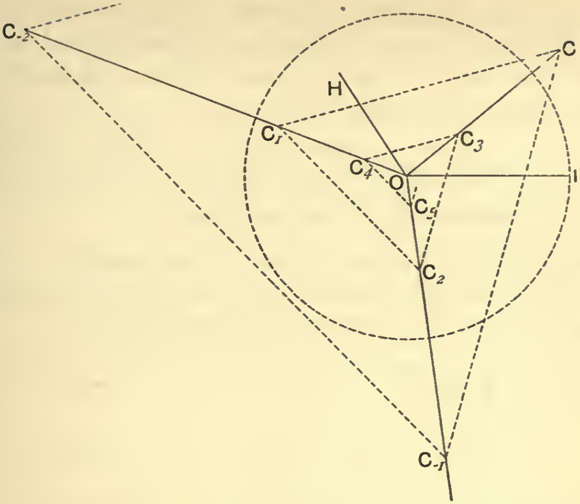


(2) If $m=0$ or an integer, the common vectorial ratio reduces to η^{-4n}, a scalar, and therefore all the vectors of the series coincide in direction, their tensors being in geometrical progression, as in the diagram annexed, in which

$$\overline{OC} = \eta^{-nu}\, i^{n \log_\eta r}, \; \overline{OH} = \eta^{-4n}.$$



(3) If $m=\frac{p}{q}$, where p is prime to q, the vectors have their directions coincident in succession with q distinct directions, each inclined to the next at an angle of $\frac{4}{q}$ right angles. This is illustrated in the annexed diagram for the case where $m = \frac{1}{3}$, and $OH = \frac{5}{7}$, so that $\overline{OH} = \frac{5}{7} i^{\frac{4}{3}}$.



From the foregoing discussion and illustrations it will be seen that the several values of A^B may be regarded as the vector *ribs of a fan;* which, when $m = 0$, is closed with the ribs superposed, but, as m increases, is spread out more and more, the angle between each rib and the next being the same and depending on m only; while the lengths of consecutive ribs form a geometric progression, of which the common ratio depends on n only.

12. *General Theory of Logarithms.*

The complete interpretation of a vector power of a vector involves also a complete extension of the Theory of Logarithms.

It will be well however to investigate this theory independently.

The definition of a logarithm is that

"If $A^B = C$, then B is the logarithm of C to the base A,"

and this remains the definition when A, B, C (some or all of them) denote vectors.

It will be found convenient in the sequel to confine the term *logarithm* to the logarithm of a *number* to a *numerical* base, and to use the term *logometer* (introduced by De Morgan) for the general logarithm, denoting the former by the usual syllable *log* and the latter by the symbol λ.

We have already seen that $\eta^i = i$, where $\eta = e^{\frac{\pi}{2}} = 4.810475\ldots$, therefore by definition $\lambda_\eta i = i$ and $\lambda_e i = \frac{\pi}{2} i$.

From this result the general theory readily follows.

The known laws of logarithms of course hold for logometers, so that we may state the following fundamental laws :

$$\lambda (AB) = \lambda A + \lambda B, \qquad \lambda \left(\frac{A}{B}\right) = \lambda A - \lambda B, \qquad \lambda (A^B) = B \lambda A,$$

$$\lambda_B A = \frac{\lambda_C A}{\lambda_C B}.$$

13. *Logometers to base* η.

Let us first consider the logometer to the numerical base η of the vector ri^u.

Since $$ri^u = r\eta^{iu}$$

$$\lambda_\eta (ri^u) = \log_\eta r + iu = r' i^{u'}, \text{ if } r' = \left\{(\log_\eta r)^2 + u^2\right\}^{\frac{1}{2}}$$

$$\text{and } \tan u' = \frac{u}{\log_\eta r}$$

or, if we generalise by putting $u + 4k$ for u, (k being an integer)

$$\lambda_\eta (ri^u) = \log_\eta r + i(u + 4k) = r' i^{u'}, \text{ if } r' = \left\{(\log_\eta r)^2 + (u + 4k)^2\right\}^{\frac{1}{2}}$$

$$\text{and } \tan u' = \frac{u + 4k}{\log_\eta r}.$$

Hence the logometer is any one of a series of vectors, all having the same *project* $\log_\eta r$, but whose *trajects* are the terms of an arithmetic progression continued upwards and downwards from u (or as many units of length as there are right angles in the inclination of the vector) with the common difference 4.

The same result may be stated thus:

The logometer of a vector (ri^u) to the base η has an infinite number of values, forming a vectorial arithmetic progression with the common difference $4i$, and extending upwards and downwards from a fundamental vectorial value defined by the project, $\log_\eta r$ or the logarithm of its tensor and the traject, (u) equal in numerical value to the number of right angles in its inclination.

It is plain that the logometer can only be scalar, if $u=0$ or a multiple of 4 right angles, that is, if the original vector is a scalar.

14. *Logometers to any numerical base.*

Let ζ denote any numerical base,

then $$\lambda_\zeta(ri^u)=\frac{\lambda_\eta(ri^u)}{\log_\eta\zeta}=\frac{\log_\eta r}{\log_\eta\zeta}+i\frac{u+4k}{\log_\eta\zeta}=\log_\zeta r+i\frac{u+4k}{\log_\eta\zeta}\cdot$$

In the particular case where $\zeta=e$,

since $$\eta=e^{\frac{\pi}{2}}\text{ so that }1=\frac{\pi}{2}\log_\eta e\text{ or }\frac{1}{\log_\eta e}=\frac{\pi}{2},$$

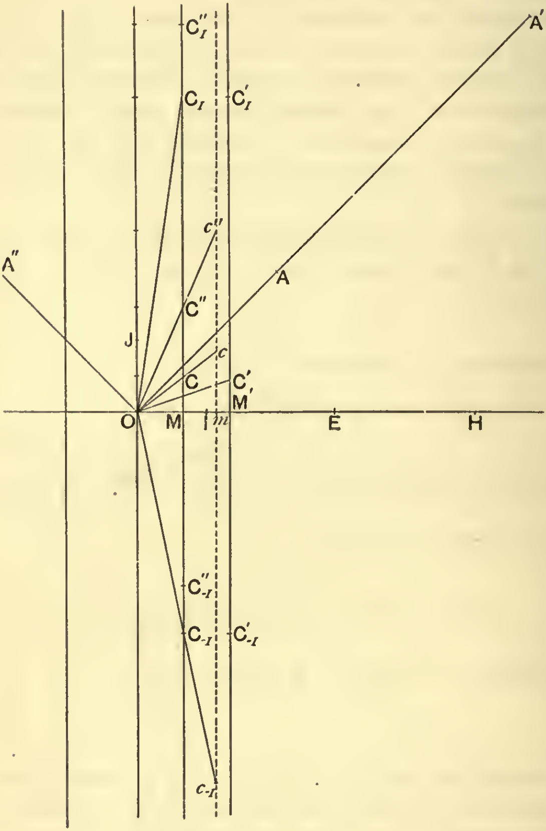
if the inclination in radians $=\theta$, so that $\theta=\frac{\pi}{2}u$,

$$\lambda_e(ri^u)=\lambda_e(r\epsilon^{i\theta})=\log_e r+i(\theta+2k\pi).$$

The change from the base η to the base ζ, being effected by the multiplication by the scalar factor $1/\log_\eta\zeta$, alters the tensors of the logarithms only in a given ratio without changing their inclination.

15. *Illustrative Diagram.*

The foregoing results are illustrated in the annexed diagram, in which OA representing a given vector, OA' is another vector having the same inclination, and $\overline{OA''}$ another having the same tensor.



Then OI, OE, OH being 1, e, η respectively,

if $\quad OM = \log_\eta OA$ or $\log_\eta OA''$, $\;Om = \frac{\pi}{2} \cdot OM = \log_e OA$ or $\log_e OA''$,

$$OM' = \log_\eta OA'$$

and $\overline{MC} = \overline{M'C'} = i \times$ measure of IOA in right angles,

$$\overline{MC''} = i \times \ \ldots\ldots \ IOA'' \ \ldots\ldots$$

$$\overline{mc} = i \times \ \ldots\ldots \ IOA \text{ in radians,}$$

then $\overline{OC}$, $\overline{OC'}$, $\overline{OC''}$ are the fundamental logometers to the base η of $\overline{OA}$, $\overline{OA'}$, $\overline{OA''}$ respectively, and $\overline{Oc}$, $\overline{Oc''}$, those of $\overline{OA}$, $\overline{OA''}$ to the base e.

Farther if

$$\overline{CC_1} = \overline{C_1C_2} = \ldots = 4i, \text{ and } \overline{CC_{-1}} = \overline{C_{-1}C_{-2}} = \ldots = -4i,$$

the values of $\lambda_\eta \overline{OA}$ are the infinite series of vectors

$$\ldots \overline{OC_{-2}}, \overline{OC_{-1}}, \overline{OC}, \overline{OC_1}, \overline{OC_2} \ldots$$

If the line MCC_1 is shifted so that

$$M \text{ comes to } M', C \text{ to } C', C_1 \text{ to } C'_1, \text{\&c.,}$$

the values of $\lambda_\eta \overline{OA'}$ are the infinite series of vectors

$$\ldots \overline{OC'''_{-2}}, \overline{OC'_{-1}}, \overline{OC'}, \overline{OC'_1} \ \overline{OC'_2} \ldots$$

If all the points in MCC_1 are shifted along the line through the same distance (namely, $i \times$ measure of AOA'' in right angles), so that C comes to C'', C_1 to C''_1, &c., the values of $\lambda_\eta \overline{OA''}$ are the infinite series of veetors

$$\ldots \overline{OC''_{-2}}, \overline{OC''_{-1}}, \overline{OC''}, \overline{OC''_1}, \overline{OC''_2} \ \ldots$$

If the vectors of the points in MCC_1 are extended to the perpendicular through m, so that $\ldots C_{-1}, C, C_1 \ldots$ come to $\ldots c_{-1}\ c\ c_1 \ldots$, the values of $\lambda_e \overline{OA}$ are the infinite series of vectors $\ldots Oc_{-1}, Oc, Oc_1 \ldots$

and those of $\lambda_e OA''$ are in like manner

$$\ldots Oc''_{-1}, Oc'', OC_1'' \ldots$$

Observe that if the inclination of $\overline{OA}$ is given, according as OA is greater or less than OH, OM is greater or less than OI: if $OA = OE$, m coincides with I: if $OA = OI$, both M and m coincide with O, and the logometers to every base coincide in direction with i: if OA is less than OI, OM is negative. In all these changes the C's move on lines parallel to OI.

Ex. If $\overline{OD}$, $\overline{OD'}$, $\overline{OD''}$ . . . are the logometers of $\overline{OC}$, the fundamental value of A^B, and through D, D', D'' . . . lines are drawn perpendicular to the vector B (or $m+ni$) intersecting lines parallel to $DD'D''$ drawn through M, M_1, M_2 . . . points on the OI axis separated by the common interval $-4n$, OM being equal to log OC: then the vectors drawn from O to the intersections of these two systems of lines are all values of $\lambda(A^B)$.

16. *Logometers to a vector base.*

The conception of a logarithm may be still further generalized by taking for the base a vector, instead of (as hitherto) a purely numerical quantity.

Let us examine the logometer of R $(=ri^u)$ to the base A $(=ai^q)$. Then since $\lambda_\eta R = \log_\eta r + i(u+4k)$ and $\lambda_\eta A = \log_\eta u + i(q+4k')$,

$$\lambda_A R = \frac{\lambda_\eta R}{\lambda_\eta A} = \frac{\log_\eta r + i(u+4k)}{\log_\eta a + i(q+4k')},$$

or, if α, θ are the circular measures of $q^{\llcorner}$, $u^{\llcorner}$ respectively,

$$\lambda_A R = \frac{\log_e r + i(\theta + 2k\pi)}{\log_e a + i(\alpha + 2k'\pi)},$$

where k, k' are unconnected integers.

The logometers therefore form an infinito-infinite series of vectors, each being the quotient of some one of the logometers of R to the base η by some one of those of A to the same base.

The meaning of this result will be made clearer by its diagrammatic representation in Figs. 1 & 2.

Let . . . . $\overline{OA_{-1}}$, $\overline{OA}$, $\overline{OA_1}$ . . . be the vectors of the series $\lambda_\eta A$,

and . . . . $\overline{OR_{-1}}$, $\overline{OR}$, $\overline{OR_1}$ . . . those of the series $\lambda_\eta R$,

the points . . . . A_{-1}, A, A_1 . . . lying on the perpendicular to OI through M, where $OM = \log_\eta a$, while . . . R_{-1}, R, R_1 . . . lie on that through N, where $ON = \log_\eta r$. To form the quotients of the vectors of the latter series divided by $\overline{OA}$, we must divide each tensor by the tensor OA and turn it negatively through the angle equal to AOI. The first operation is effected for all the vectors by shifting the line of R's parallel to itself till N comes to N', such that $ON' = ON/OA$ or $ON'/ON = OI/OA$; then,

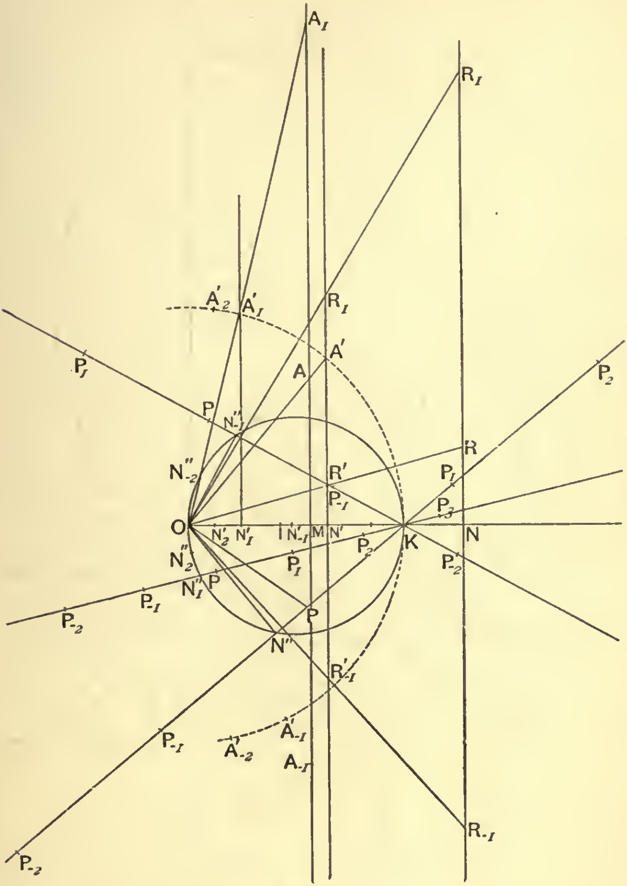


FIG. 1.

if OA meet the line of R's in this new position in A', the second operation is effected by turning the triangle $OA'N'$ about O until OA' coincides with OI. If then A' comes to K and N' to N'', the line of R's finally coincides with the infinitely extended line through K and N''.

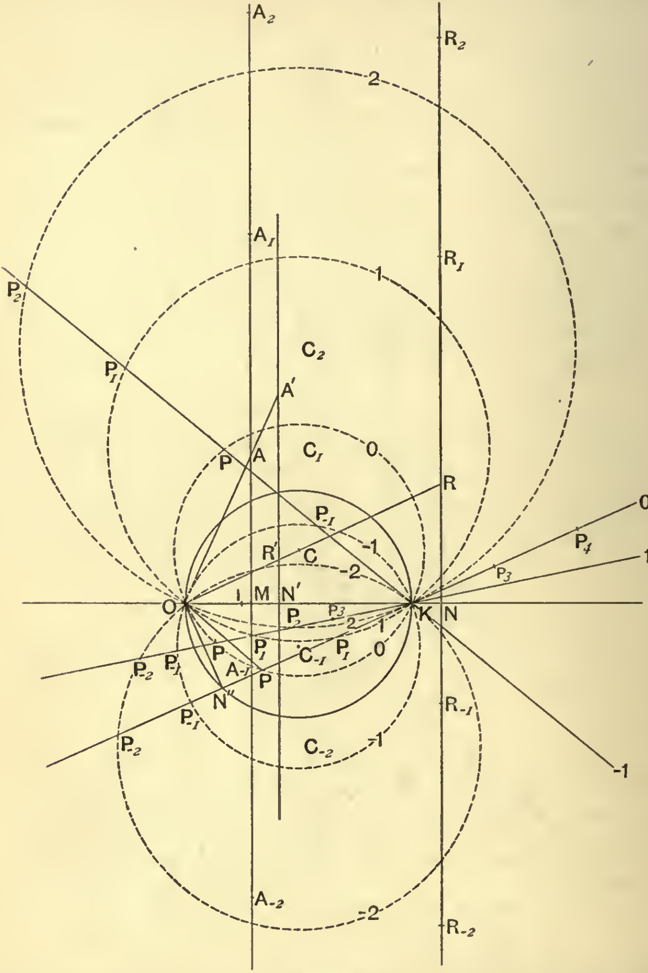

A₂
R₂
2
A₁
R₁
1
P₂
C₂
P₁
A′
C₁
0
P
A
R
P₁
0
P₄
R′
C
-1
-2
1
P₃
1
O
M
N′
K
N
P₂
P₃
2
1
-1
P
P₁
C₋₁
P₁
0
P₂
P₁
A₋₁
P
N″
R₋₁
P₋₁
P₂
C₋₂
-1
A₋₂
-2
R₋₂

FIG. 2.

Now $OK=OA'=OA.ON'/OM=OI.ON/OM$, since $ON'.OA=ON.OI$ from above, therefore

$$OK=\frac{\log_\eta r}{\log_\eta a}=\log_a r,$$

and since $ON''K$ is a right angle, N'' is on the circumference of the circle, diameter OK.

Hence one series of values of $\lambda_A R$ is found thus. On OI take $OK=\log_a r$, and on OK as diameter describe a circle, draw ON'', to meet the circumference in N'', making the angle $KON''=AON$, then if $\dots P_{-1}, P, P_1 \dots$ are taken on $N''K$ so that $\dots N''P_{-1}, N''P, N''P_1 \dots$ are equal respectively to $\dots N'R'_{-1}, N'R', N'R'_1, \dots$ the series of vectors $\overline{OP}_{-1}, \overline{OP}, \overline{OP}_1, \dots$ form one series of values of $\lambda_A R$.

In like manner, if $N''_1, N''_{-1}, N''_2, N''_{-2}, \dots$ are points on the circumference, such that the angles $KON''_1, KON''_{-1}, \dots$ are equal to $A_1ON, A_{-1}ON \dots$ respectively, there will be like series of vectors drawn to the R's on each of the lines $KN''_1, KN''_{-1}, KN''_2, KN''_{-2}, \dots$ forming for each a series of values of $\lambda_A R$.

Hence while logometers to a numerical base are vectors drawn from O to points distributed at equal intervals along one line perpendicular to OI, logometers to a vector base are vectors drawn from O to points similarly distributed along each of a pencil of lines all passing through one point K on OI, such that $OK=$ the logarithm of the tensor of the given vector to the tensor of the given vector-base as base. Farther it is to be observed that (as shown in Fig. 2) all the points P, corresponding to the logometers of $\overline{OR}$ to the several bases $\dots \overline{OA}_{-1}, \overline{OA}, \overline{OA}_1 \dots$ lie on a circle through O, K divided by OK into segments containing angles equal to the supplementary angles at P. Similarly the points $P_1, P_2 \dots$ lie respectively on circles divided by OK into segments containing angles equal to the supplementary angles at R. The centres of these circles are the intersections of the line bisecting OK at right angles with the lines $\dots OR_{-1}, OR, OR_1 \dots$

Hence the complete series of values of $\lambda_A R$ is the series of vectors drawn from O to the intersections of the pencil of lines through K corresponding to $\dots A_{-1}, A, A_1, \dots$ with the circles through O, K corresponding to $\dots R_{-1}, R, R_1 \dots$

CHAPTER VI

EXCIRCULAR OR HYPERBOLIC TRIGONOMETRY

1. THE measure of an angle is essentially scalar, but the assumption of a vector form for such measure leads to a series of functions which have a close analogy to the trigonometrical functions and which have relations to the rectangular * hyperbola corresponding very closely with those of the trigonometrical functions to the circle. A discussion of these functions is necessary to give completeness to our subject, and will be found useful in its applications to higher Analysis, particularly in the integration of certain algebraical forms.

2. Returning to the formulæ (Chap. V. § 9), u, θ being the measures of an angle in right angles and radians respectively,

$$\cos u = \frac{i^u + i^{-u}}{2} = \frac{\eta^{iu} + y^{-iu}}{2} = \frac{\epsilon^{i\theta} + \epsilon^{-i\theta}}{2} = \cos\ \theta$$

$$\sin u = \frac{i^u - i^{-u}}{2i} = \frac{\eta^{iu} - \eta^{-iu}}{2i} = \frac{\epsilon^{i\theta} - \epsilon^{-i\theta}}{2i} = \sin\ \theta.$$

Put iu for u, or $i\theta$ for θ, then

$$\cos iu = \frac{\eta^{i^2u} + \eta^{-i^2u}}{2} = \frac{\eta^{-u} + \eta^{u}}{2} \text{ or } \cos i\theta = \frac{\epsilon^{i^2\theta} + \epsilon^{-i^2\theta}}{2} = \frac{\epsilon^{-\theta} + \epsilon^{\theta}}{2}$$

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* Observe that these functions have relation not to the hyperbola generally, but to the *rectangular* hyperbola, which is among hyperbolas what the circle is among ellipses. It would be impossible to use the octosyllabic, though correct, term "rectangular hyperbolic" functions; I propose, therefore, to call the "rectangular hyperbola" the "excircle," and its functions "excircular."

and

$$\sin iu = \frac{\eta^{i^2u} - \eta^{-i^2u}}{2i} = -\frac{\eta^u - \eta^{-u}}{2i} = i\,\frac{\eta^u - \eta^{-u}}{2}$$

$$\text{or } \sin\theta = i\,\frac{\epsilon^\theta - \epsilon^{-\theta}}{2}.$$

The quantities

$$\frac{\eta^u + \eta^{-u}}{2},\ \frac{\eta^u - \eta^{-u}}{2} \text{ or } \frac{\epsilon^\theta + \epsilon^{-\theta}}{2},\ \frac{\epsilon^\theta - \epsilon^{-\theta}}{2}$$

are termed respectively the *hyperbolic cosine* and *hyperbolic sine* of the angle, whose measure in right angles is u and in radians θ, and they are usually denoted as cosh u, sinh u. The rectangular hyperbola, from the point of view in which we shall have to regard it in the sequel, would be not inappropriately termed an *excircle*, and these functions the *excosine* and *exsine* respectively, and as these terms have the advantage of brevity in speech, we shall generally make use of them. The *notation* however, *cosh*, *sinh*, *tanh*, is so well established, that I have not ventured to substitute for these *excos*, *exsin*, *extan*, or *ecos*, *esin*, *ectan*, as it otherwise would have been natural to do.*

Hence we have

$$\frac{\eta^u + \eta^{-u}}{2} = \cosh u = \cos iu \text{ and}$$

$$\frac{\eta^{iu} + \eta^{-iu}}{2} = \cosh iu = \cos u,$$

$$\frac{\eta^u - \eta^{-u}}{2} = \sinh u = -i \sin iu \text{ and}$$

$$\frac{\eta^{iu} - \eta^{-iu}}{2} = \sinh iu = i \sin u,$$

and therefore

$$\cosh u + \sinh u = \eta^u, \quad \text{as} \quad \cos u + i \sin u = \eta^{iu}$$

$$\cosh u - \sinh u = \eta^{-u}, \quad \text{as} \quad \cos u - i \sin u = \eta^{-iu}$$

---

* It will be found however convenient to *read* at any rate the unpronounceable syllables *sinh*, *tanh*, as *ecsin*, *extan*.

so that $\cosh^2 u - \sinh^2 u = 1,$ as $\cos^2 u + \sin^2 u = 1,$

and $(\cosh u \pm \sinh u)^n = \eta^{nu} = \cosh nu \pm \sinh nu,$

as $(\cos u \pm i \sin u)^n = \cos nu \pm i \sin nu.$

Farther, if we define *tanh*, *coth*, *sech*, *cosech* similarly to *tan*, *cot*, *sec*, *cosec*,

$$\tanh u = -i \tan iu = \frac{\eta^u - \eta^{-u}}{\eta^u + \eta^{-u}} \qquad i \tan u = \tanh iu = \frac{\eta^{iu} - \eta^{-iu}}{\eta^{iu} + \eta^{-iu}}$$

$$\coth u = -i \cot iu \qquad \cot u = \coth iu$$

$$\operatorname{sech} u = \sec iu \qquad \sec u = \operatorname{sech} iu$$

$$\operatorname{cosech} u = -i \operatorname{cosec} iu \qquad \operatorname{cosec} u = i \operatorname{cosech} iu.$$

We have also the inverse forms of the above formulae, as

$$\cosh^{-1} x = -i \cos^{-1} x \qquad \cos^{-1} x = i \cosh^{-1} x$$

and the like, which the student should write down for each function.

Also from $$x = \cosh u = \frac{\eta^u + \eta^{-u}}{2},$$

$$\eta^u = x \pm \sqrt{x^2 - 1} = x + \sqrt{x^2 - 1} \text{ or } \frac{1}{x + \sqrt{x^2 - 1}}$$

whence $u = \lambda_\eta(x \pm \sqrt{x^2 - 1}) = \pm \log_\eta(x + \sqrt{x^2 - 1}) + 4ki,$
so that, if $u_0 = \log_\eta(x + \sqrt{x^2 - 1})$, and writing Cosh with a capital C to denote the particular angle u_0,

$$u = \cosh^{-1} x = \pm u_0 + 4ki = \pm \operatorname{Cosh}^{-1} x + 4ki.$$

Hence $\cosh^{-1} x$ has the same multiplicity of values that $\cos^{-1} x$ has, but with the difference that, while in the case of $\cos^{-1} x$ all the values (if any) are scalar, in the case of $\cosh^{-1} x$ two only (if any) are scalar.

A like remark applies to the other inverse functions. The comparative formulae (which the student should verify) are

$$\mathrm{Cosh}^{-1}x = \log_\eta(x + \sqrt{x^2 - 1})$$

and $\cosh^{-1}x = \pm\,\mathrm{Cosh}^{-1}x + 4ki$, as $\cos^{-1}x = \pm\,\mathrm{Cos}^{-1}x + 4k$.

$$\mathrm{Sinh}^{-1}x = \log_\eta(x + \sqrt{x^2 + 1})$$

and $\sinh^{-1}x = (-1)^k\mathrm{Sinh}^{-1}x + 2ki$, as $\sin^{-1}x = (-1)^k\mathrm{Sin}^{-1}x + 2k$.

$$\mathrm{Tanh}^{-1}x = \tfrac{1}{2}\log_\eta\frac{1+x}{1-x}$$

and $\tanh^{-1}x = \mathrm{Tanh}^{-1}x + 2ki$, as $\tan^{-1}x = \mathrm{Tan}^{-1}x + 2k$.

If we use circular measure, the first becomes

$\mathrm{Cosh}^{-1}x = \log_e(x + \sqrt{x^2 - 1})$, $\cosh^{-1}x = \pm\ \mathrm{Cosh}^{-1}x + 2k\pi i$

as $\qquad \cos^{-1}x = \pm\,\mathrm{Cos}^{-1}x + 2k\pi$,

and similarly for the others.

3. *Range of the values of cosh, sinh, tanh. Hyperbolic Amplitude.*

Since $\cosh u = \dfrac{\eta^u + \eta^{-u}}{2}$, so long as u is scalar, $\cosh u$ has a positive scalar value, not less than 1, or it ranges from $+1$ to $+\infty$.

Again $\sinh u = \dfrac{\eta^u - \eta^{-u}}{2}$, and therefore, u being scalar, $\sinh u$ has any scalar value whatever, positive or negative, or it ranges from $-\infty$ to $+\infty$.

Also $\tanh u = \dfrac{\eta^u - \eta^{-u}}{\eta^u + \eta^{-u}} = \dfrac{\eta^{2u} - 1}{\eta^{2u} + 1}$, and therefore, u being scalar, $\tanh u$ has any scalar value between the limits -1 and $+1$.

Let τ denote an angle between the limits $-1^{\llcorner}$ and $+1^{\llcorner}$; then consistently with the above we might assume

$$\sec\tau = \cosh u \ \text{ or } \ \tan\tau = \sinh u \ \text{ or } \ \sin\tau = \tanh u,$$

but, by the relations

$$\cosh^2 u - \sinh^2 u = 1 \text{ compared with } \sec^2 \tau - \tan^2 \tau = 1$$

and $$\tanh u = \frac{\sinh u}{\cosh u} \text{ with } \sin \tau = \frac{\tan \tau}{\sec \tau},$$

the last two follow from the first, so that for the same angle τ we have

$$\sec \tau = \cosh u \qquad \tan \tau = \sinh u \qquad \sin \tau = \tanh u$$

and $$\cos \tau = \operatorname{sech} u \qquad \cot \tau = \operatorname{cosech} u \qquad \operatorname{cosec} \tau = \coth u.$$

The angle τ has been termed the *Hyperbolic Amplitude* of u, and denoted as amh u.

Since

$$\sin \tau = \tanh u = \frac{\eta^{2u} - 1}{\eta^{2u} + 1}, \quad \eta^{2u} = \frac{1 + \sin \tau}{1 - \sin \tau} = \tan^2 \left(\frac{1}{2} + \frac{\tau}{2}\right),$$

therefore

$$\operatorname{amh} u = \tau = \operatorname{Sin}^{-1} \frac{\eta^{2u} - 1}{\eta^{2u} + 1}, \quad u = \tfrac{1}{2} \log_\eta \frac{1 + \sin \tau}{1 - \sin \tau} = \log_\eta \tan\left(\frac{1}{2} + \frac{}{2}\right);$$

or, using radians,

$$\operatorname{amh} \theta = \operatorname{Sin}^{-1} \frac{e^{2\theta} - 1}{e^{2\theta} + 1}, \quad \theta = \tfrac{1}{2} \log_e \frac{1 + \sin \tau}{1 - \sin \tau} = \log_e \tan\left(\frac{\pi}{4} + \frac{\tau}{2}\right).$$

It will be observed that, while for scalar values of the angle, the circular functions *sin*, *cos*, *tan* are periodic, the excircular functions sinh, cosh, tanh are non-periodic. This fact corresponds to the limitation of the hyperbolic amplitude to be a positive or negative acute angle.

4. *Formulae for excircular functions.*

The formulae established in previous chapter for circular functions may, by means of the relations in § 2, and putting iu for u, be transformed into formulae for excircular functions. The following list contains the principal formulae which it is desirable to remember.

The formulae below

For Circular Functions *become* *for Excircular Functions,*

$$\cos^2 u + \sin^2 u = 1, \qquad \cosh^2 u - \sinh^2 u = 1,$$

$$1 + \tan^2 u = \sec^2 u, \qquad 1 - \tanh^2 u = \operatorname{sech}^2 u,$$

$$\cos (u \pm v) = \cos u \cos v \mp \sin u \sin v,$$

$$\cosh (u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v,$$

$$\sin (u \pm v) = \sin u \cos v \pm \cos u \sin v,$$

$$\sinh (u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v,$$

$$\sin 2u = 2 \sin u \cos u, \qquad \sinh 2u = 2 \sinh u \cosh u,$$

$$\cos 2 u = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u,$$

$$\cosh 2u = \cosh^2 u + \sinh^2 u = 2 \cosh^2 u - 1 = 1 + 2 \sinh^2 u,$$

$$\tan (u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \,.\, \tan v}, \qquad \tanh (u \pm v) = \frac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v},$$

$$\tan (u + v + w + \ldots) = \frac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 + \ldots},$$

$$\tanh (u + v + w + \ldots) = \frac{S_1 + S_3 + S_5 + \ldots}{1 + S_2 + S_4 + \ldots},$$

$$\cos u + \cos v = 2 \cos \frac{u+v}{2} \,.\, \cos \frac{u-v}{2},$$

$$\cosh u + \cosh v = 2 \cosh \frac{u+v}{2} \,.\, \cosh \frac{u-v}{2},$$

$$\sin u + \sin v = 2 \sin \frac{u+v}{2} \,.\, \cos \frac{u-v}{2},$$

$$\sinh u + \sinh v = 2 \sinh \frac{u+v}{2} \,.\, \cosh \frac{u-v}{2},$$

$$\cos u - \cos v = 2 \sin \frac{u+v}{2} \,.\, \sin \frac{v-u}{2},$$

$$\cosh u - \cosh v = 2 \sinh \frac{u+v}{2} \,.\, \sinh \frac{u-v}{2},$$

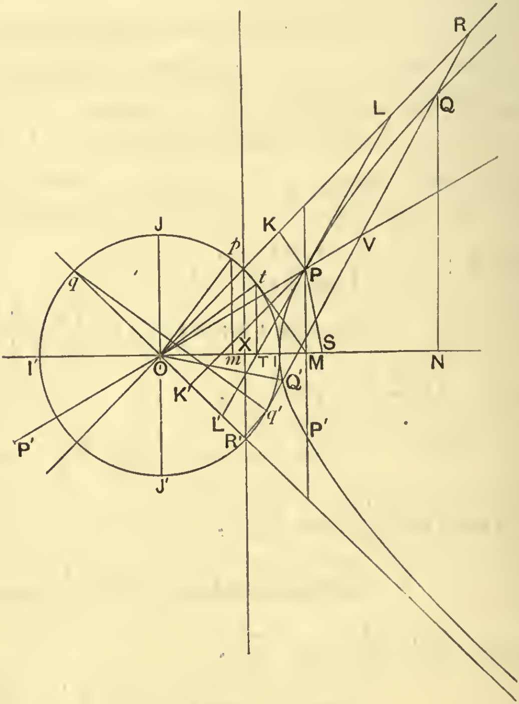
$$\sin u - \sin v = 2 \cos \frac{u+v}{2} \,.\, \sin \frac{u-v}{2},$$

$$\sinh u - \sinh v = 2 \cosh \frac{u+v}{2} \,.\, \sinh \frac{u-v}{2}.$$

It will be instructive for the student to obtain these formulae independently from the definitions of the excircular functions.

5. *Geometrical Interpretation of Excircular Functions.*

On the unit circle IJI' take the arc $Ip = u$ and the arc $It = \tau$ or amh u: draw pm, tI perpendicular to OI, and tM, the tangent to the unit circle at t.



Then $\qquad OM = \sec\tau = \cosh u$ and $Mt = \tan\tau = \sinh u$;

hence, if PMP' is drawn perpendicular to OI, making

$$MP = MP' = Mt,$$

P will correspond to p, so that, as Om, mp are $\cos u$, $\sin u$ respectively, so OM, MP, are $\cosh u$, $\sinh u$ respectively, and OM, MP' are $\cosh(-u)$, $\sinh(-u)$ respectively. And, since $\cosh^2 u - \sinh^2 u = 1$, $OM^2 - MP^2 = 1$, so that, while the locus of p is a circle, the locus of P, which may appropriately be termed the "*excircle*," is the rectangular hyperbola, whose semi-axis is OI; and, as the vector equation to the circle is

$$\overline{Op} = \cos u + i \sin u,$$

that to the hyperbola is

$$OP = \cosh u + i \sinh u.$$

We proceed to show that, if P, Q are points on the excircle corresponding to p, q on the circle, the areas of the excircular or hyperbolic sector POQ and the circular sector pOq are equal.

Let the angle $IOq = v$ and let QN be the ordinate of Q, then the $\triangle\, POQ = \triangle OQN - \triangle OPN -$ trapezium $PMNQ$,

$$\begin{aligned} &= \tfrac{1}{2}\left\{ON \,.\, NQ - OM \,.\, MP - (MP + NQ)(ON - OM)\right\}, \\ &= \tfrac{1}{2}\left\{OM \,.\, NQ - ON \,.\, MP\right\}, \\ &= \tfrac{1}{2}(\cosh u \sinh v - \cosh v \sinh u), \\ &= \tfrac{1}{2}\sinh(v-u), \text{ from § 4,} \end{aligned}$$

and the triangle $pOq = \frac{1}{2}\sin(v-u)$;

so that $\dfrac{\text{triangle } POQ}{\text{triangle } pOq} = \dfrac{\sinh(v-u)}{\sin(v-u)}.$

But, since

$$\frac{\sinh u}{u} = \frac{\eta^u - \eta^{-u}}{2u} = \eta^{-u} \,.\, \frac{\eta^{2u} - 1}{2u},$$

$\text{limit}_{\,u=0} \dfrac{\sinh u}{u} = 1 \,.\, \text{lt.}_{\,u=0} \dfrac{\eta^{2u}-1}{2u} = \dfrac{\pi}{2}$, and $\text{lt.}_{\,u=0} \dfrac{\sin u}{u} = \dfrac{\pi}{2}$,

therefore

$$\text{lt.}_{\,u=0} \frac{\sinh u}{\sin u} = \frac{\text{lt.}_{\,u=0} \dfrac{\sinh u}{u}}{\text{lt.}_{\,u=0} \dfrac{\sin u}{u}} = 1$$

and

$$\text{limit}_{\,v=u} \frac{\sinh(v-u)}{\sin(v-u)} = 1,$$

so that the triangles POQ, pOq are ultimately equal, when Q, q come to coincide with P, p.

Hence since two finite sectors POQ, pOq can be divided into the same number of elementary sectors, each corresponding pair of which are ultimately equal, the limits of the sums of the elements, or the sectors themselves, are equal.

Instead of regarding $\cos u$, $\sin u$, &c., as functions of the *angle* IOp we might regard them as functions of the corresponding *sector* of the unit-circle, and then regarding $\cosh u$, $\sinh u$, &c., in like manner as functions of the excircular or hyperbolic sector IOP, the analogy between the circular and excircular functions is complete.

The sector $IOP =$ sector $IOp = u$ quadrants of the unit circle

$$= \frac{\pi}{4} u \text{ or } \tfrac{1}{2}\theta \text{ square units.}$$

If the angle $IOP = \phi$ radians,

$$\tan\phi = \tanh\theta = \frac{e^{2\theta}-1}{e^{2\theta}+1}$$

so that

$$\theta = \tfrac{1}{2}\log_e \frac{1+\tan\phi}{1-\tan\phi} = \tfrac{1}{2}\log_e \left(\tan\frac{\pi}{4}+\phi\right),$$

therefore the area of the sector IOP in terms of ϕ is

$$\tfrac{1}{4}\log_e \tan\left(\frac{\pi}{4}+\phi\right) \text{ square units.}$$

6. If IOP_1, P_1OP_2, P_2OP_3 . . . are the hyperbolic sectors corresponding to the successive quadrants of the circle, and M_1, M_2 . . . the feet of the ordinates to P_1, P_2 . . ., $OM_1 = \cosh 1^{L} = 2.5$, $OM_2 = \cosh 2^{L} = 11.5$, $OM_3 = \cosh 3^{L} = 55.5$, $OM_4 = \cosh 4^{L} = 245.5$, approximately. These values show the rapidity with which the arcs IP_1, P_1P_2, P_2P_3 . . ., corresponding to successive quadrantal arcs of the circle, increase. The successive sectors, corresponding to successive complete revolutions of a radius of the circle, form a series of more and more elongated sectors with smaller and smaller angles, of which no finite number will fill up

the space between the curve and its asymptotes, so that this area is infinite.

Riemann, in his researches on the complex variable, found it convenient to conceive a revolving radius after completing each revolution to continue its path, not on the same plane, but on a plane superposed upon the previous one, thus moving on a continuous surface without passing through the same position a second time. In the present case, if we suppose an infinite number of circles superposed on the unit circle above and below it and all of them cut along the radius OI', and then each terminal radius of the one (the revolution being supposed in the positive direction) united with the coincident initial radius of the next above it, we shall have a Riemann's surface (a screw surface with a zero pitch), whose folds, infinite in number, correspond to the successive sectors, also infinite in number and each equal to the area of the unit circle, which fill up the area between the excircle and its asymptotes.

It will be observed that it is only one branch of the hyperbola that is thus determined as corresponding to the complete Riemann's surface, as appears from the fact that $\cosh u$ is for scalar values of u essentially positive. To obtain the other branch, we must take vector values $u + i$ or $u + 2i$. The former gives

$$\cos(u + i) + i \sinh(u + i) = -\cosh u + i \sinh u,$$

which gives the second branch as the *image* of the first with respect to the OJ axis.

The latter gives

$$\cosh(u + 2i) + i \sinh(u + 2i) = -(\cosh u + i \sinh u)$$

which gives the second branch by turning the first through two right angles.

7. *Properties of the Excircle deduced from Excircular Functions.*

It is instructive to see the manner in which many of the well-known properties of the hyperbola are at once deducible from the formulae of excircular functions. The following will serve as illustrations and suggestions for the mode of treatment of similar geometrical investigations.

The vector equation to the excircle,

$$\overline{OP} = \cosh u + i \sinh u,$$

becomes by substituting the exponential expressions for $\cosh u$, $\sinh u$,

$$\overline{OP}=\frac{\eta^u+\eta^{-u}}{2}+i\frac{\eta^u-\eta^{-u}}{2}=\frac{\eta^u}{\sqrt{2}}\frac{1+i}{\sqrt{2}}+\frac{\eta^{-u}}{\sqrt{2}}\frac{1-i}{\sqrt{2}}$$

$$=\frac{\eta^u}{\sqrt{2}}i^{\frac{1}{2}}+\frac{\eta^{-u}}{\sqrt{2}}i^{-\frac{1}{2}},$$

since

$$i^{\pm\frac{1}{2}}=\cos\tfrac{1}{2}\pm i\sin\tfrac{1}{2}=\frac{1\pm 1}{\sqrt{2}}.$$

Take $\overline{OK}=\frac{\eta^u}{\sqrt{2}}i^{\frac{1}{2}}$, that is, a vector whose tensor is $\eta^u/\sqrt{2}$ and whose inclination to $\overline{OI}$ is half a right angle, and $\overline{OK'}=\frac{\eta^{-u}}{\sqrt{2}}i^{-\frac{1}{2}}$, that is, a vector whose tensor is $\eta^{-u}/\sqrt{2}$ and inclination half a right angle negatively: then

$$\overline{OP}=\overline{OK}+\overline{OK'}$$

and OP is a diagonal of the rectangle $OKPK'$, whose area

$$=OK\,.\,OK'=\frac{\eta^u}{\sqrt{2}}\cdot\frac{\eta^{-u}}{\sqrt{2}}=\tfrac{1}{2}.$$

Observe also that, if u is positive and very large and therefore OP very large, η^{-u} is very small and PK (or OK') is very small and may be made as small as we please by taking u sufficiently large: so also if u is very large and negative, PK' is very small and diminishes without limit as u increases numerically. Hence the excircle approaches without limit to OK, or OK' as the point P is taken farther and farther from O. We have proved therefore that "the excircle has two asymptotes at right angles to one another (hence called the *Rectangular* Hyperbola), equally inclined to its axis on opposite sides of it, and that if perpendiculars are let fall from any point on the curve to the asymptotes, the area of the rectangle between these and the asymptotes is constant and equal to half the square on the semi-axis OI."

Let Q, Q' be two points on the excircle corresponding to the values $u+v$, $u-v$ of the angles IOq, IOq' on the circle, or shortly the points $(u+v)$, $(u-v)$ respectively, then

$$\overline{OQ} = \cosh(u+v) + i\sinh(u+v) \text{ or } \overline{OQ} = \frac{\eta^{u+v}}{\sqrt{2}}\, i^{\frac{1}{2}} + \frac{\eta^{-u-v}}{\sqrt{2}}\, i^{-\frac{1}{2}},$$

$$\overline{OQ'} = \cosh(u-v) + i\sinh(u-v) \text{ or } \overline{OQ'} = \frac{\eta^{u-v}}{\sqrt{2}}\, i^{\frac{1}{2}} + \frac{\eta^{-u+v}}{\sqrt{2}}\, i^{-\frac{1}{2}}.$$

If V is the middle point of the chord QQ', $\overline{OV} = \frac{1}{2}(OQ + OQ')$,

$$\therefore \overline{OV} = \tfrac{1}{2}\left(\cosh \overline{u+v} + \cosh \overline{u-v}\right) + i \,.\, \tfrac{1}{2}\left(\sinh \overline{u+v} + \sinh \overline{u-v}\right)$$

$$= \cosh v\,(\cosh u + i \sinh u) = \overline{OP} \cosh v,$$

so that the middle point of the chord QQ' lies on OP.

In another form

$$\overline{OV} = \cosh v \left(\frac{\eta^{u}}{\sqrt{2}}\, i^{\frac{1}{2}} + \frac{\eta^{-u}}{\sqrt{2}}\, i^{-\frac{1}{2}}\right).$$

Also $\overline{VQ} = \frac{1}{2}(\overline{OQ} - \overline{OQ'})$,

$$\therefore \overline{VQ} = \sinh v\,(\sinh u + i \cosh u)$$

$$= \sinh v \left(\frac{\eta^{u}}{\sqrt{2}}\, i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\, i^{-\frac{1}{2}}\right).$$

If ϕ, ψ are the inclinations of OP or $\overline{OV}$, $\overline{QQ'}$ respectively,

$$\overline{OP} = \cosh u + i \sinh u = (\cosh^2 u + \sinh^2 u)^{\frac{1}{2}} (\cos\phi + i \sin\phi)$$

$$= (\cosh 2u)^{\frac{1}{2}}\, e^{i\phi}$$

$$\overline{OV} = \overline{OP} \cosh v = (\cosh 2u)^{\frac{1}{2}} \cosh v \,.\, e^{i\phi},$$

where $\tan\phi = \tanh u$:

$$\overline{VQ} = \sinh v\,(\sinh u + i\cosh u) = \sinh v \,.\, (\cosh 2u)^{\frac{1}{2}}\, e^{i\psi} :$$

where $\tan\psi = \coth u$:

so that $\tan\phi \,.\, \tan\psi = 1$ or ϕ and ψ are complementary, and

$$OP = (\cosh 2u)^{\frac{1}{2}},\ OV = (\cosh 2u)^{\frac{1}{2}} \cosh v,\ VQ = (\cosh 2u)^{\frac{1}{2}} \sinh v.$$

Hence, if POQ, POQ' are equal sectors on opposite sides of OP or (u), QQ' has a constant inclination to the axis equal to the complement of IOP, and in all positions it is bisected by OP, and OP is therefore termed the diameter of this system of parallel chords. Also

$$OV^2 - VQ^2 = \cosh 2u\left\{\cosh^2 v - \sinh^2 v\right\} = \cosh 2u = OP^2,$$

whence

$$VQ^2 = OV^2 - OP^2 = (OV - OP)(OV + OP) = PV \,.\, P'V,$$

if P' be a point in PO produced such that $OP' = OP$, which are well-known properties of the rectangular hyperbola.

Farther, if R, R' are points on the asymptotes such that

$$\overline{OR} = \eta^u \cosh v\,\sqrt{2}\,.\,i^{\frac{1}{2}},\ \ \overline{OR'} = \eta^{-u}\cosh v\,\sqrt{2}\,.\,i^{-\frac{1}{2}},$$

$$\overline{OV} = \tfrac{1}{2}\,(\overline{OR} + \overline{OR'}),$$

and

$$\overline{VR} = \overline{OR} - \overline{OV} = \cosh v\left(\frac{\eta^u}{\sqrt{2}}\,i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\,i^{-\frac{1}{2}}\right)$$

$$= \cosh v\,(\sinh u + i\cosh u).$$

$$\overline{VR'} = \overline{OR'} - \overline{OV} = -\cosh v\left(\frac{\eta^u}{\sqrt{2}}\,i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\,i^{-\frac{1}{2}}\right)$$

$$= -\cosh v\,(\sinh u + i\cosh u).$$

Comparing this with the expression for $\overline{VQ}$, it follows that RR' coincides with QQ', and V is the middle point of both RR' and QQ', so that

$$\overline{QR} = -\overline{Q'R'} = (\cosh v - \sinh v)\left(\frac{\eta^u}{\sqrt{2}}\,i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\,i^{-\frac{1}{2}}\right)$$

$$= \eta^{-v}\,.\left(\frac{\eta^u}{\sqrt{2}}\,i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\,i^{-\frac{1}{2}}\right).$$

$$\overline{R'Q} = -\overline{RQ'} = (\cosh v + \sinh v)\left(\frac{\eta^u}{\sqrt{2}}\,i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\,i^{-\frac{1}{2}}\right)$$

$$= \eta^{v}\left(\frac{\eta^u}{\sqrt{2}}\,i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\,i^{-\frac{1}{2}}\right).$$

When $V = 0$, Q, Q' come to coincide with P, and RR' coincides with LPL', the tangent at P, and since $\cosh 0 = 1$ and $\sinh 0 = 0$,

$$\overline{L'P} = \overline{PL} = \frac{\eta^u}{\sqrt{2}}\, i^{\frac{1}{2}} - \frac{\eta^{-u}}{\sqrt{2}}\, i^{-\frac{1}{2}},\ \overline{OL} = \eta^u \sqrt{2} \,.\, i^{\frac{1}{2}} = 2\,\overline{OK},$$

$$\overline{OL'} = \eta^{-u} \sqrt{2} \,.\, i^{-\frac{1}{2}} = 2\,\overline{OK'}.$$

Hence

$$\overline{R'V} = \overline{VR} = \overline{PL} \cosh v,\ \overline{OR} = \overline{OL} \cosh v,\ \overline{OR'} = \overline{OL'} \cosh v,$$

$$\overline{VQ} = \overline{PL} \,.\, \sinh v.$$

so that the triangles OKK', OLL', ORR' are similar, with the sides KK', LL', RR' parallel and in the ratio of $1 : 2 : 2 \cosh v$, and their areas respectively $\frac{1}{4}$, 1, $\cosh^2 v$ of the unit square on OI.

Taking the tensors of $\overline{OV}$, $\overline{VR}$, $\overline{VR'}$, we have

$$OV = VR = VR' = \cosh v \,.\, (\cosh 2u)^{\frac{1}{2}},$$

and so

$$OP = PL = PL' = (\cosh 2u)^{\frac{1}{2}}.$$

Again

$$\overline{OL} \,.\, \overline{OL'} = \eta^u \sqrt{\,}.\, 2\, i^{\frac{1}{2}} \,.\, \eta^{-u} \sqrt{2} \,.\, i^{-\frac{1}{2}} = 2,$$

hence, if $\overline{OS} = \sqrt{2}$, $\overline{OL} \,.\, \overline{OL'} = \overline{OS}^2$, so that the triangles OLS, OSL' are similar, S being the point where the tangent to the circle at its intersection with either asymptote of the excircle meets the axis.

Hence

$$\overline{SP} = \overline{OP} - \overline{OS} = (\cosh u - \sqrt{2}) + i \sinh u,$$

therefore

$$\begin{aligned} SP^2 &= (\cosh u - \sqrt{2})^2 + \sinh^2 u, \\ &= \cosh^2 u - 2\sqrt{2} \cosh u + 2 + \cosh^2 u - 1, \\ &= 2\left(\cosh^2 u - \sqrt{2} \cosh u + \tfrac{1}{2}\right); \end{aligned}$$

and

$$SP = \sqrt{2}\left(\cosh u - \frac{1}{\sqrt{2}}\right) = \sqrt{2}\,(OM - OX) = \sqrt{2} \,.\, MX,$$

$$\text{if } OX = \frac{1}{\sqrt{2}},$$

so that X is the foot of the perpendicular to OI from the intersection of the circle with either asymptote.

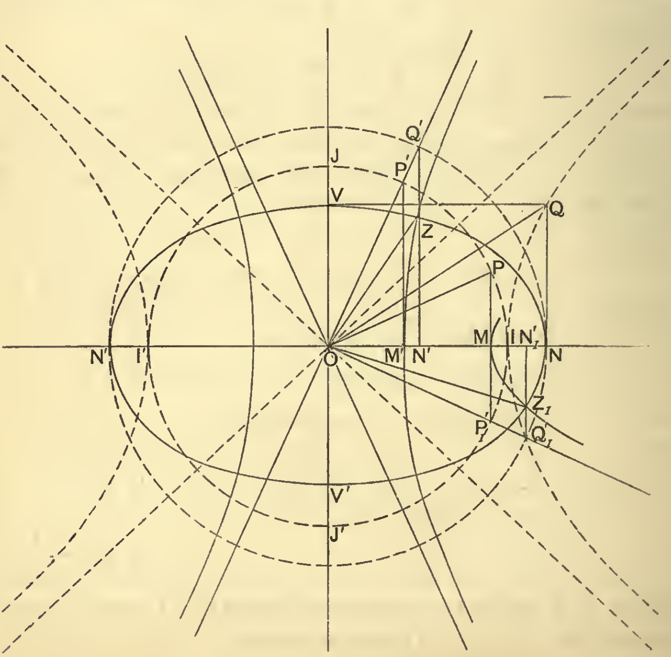
Hence the distance from P to the fixed point S (called the *focus*) has to the distance from P to this fixed perpendicular (called the *directrix*) the constant ratio $\sqrt{2} : 1$ (termed the *excentricity*), which proves the property in accordance with which the hyperbola is usually defined in elementary treatises on Conic Sections.

8. *Graphic Construction of the Vectors sin* $(u + vi)$ *and cos* $(u + vi)$.

Let $z = x + yi = \sin(u + vi)$, where x, y, u, v are scalars: then

$$x = \sin u \cosh v, \quad y = \cos u \sinh v.$$

The prime vector being OI, describe the unit circle and excircle, and let the circular sector IOP and the excircular sector IOQ represent $u^{\llcorner}$, $v^{\llcorner}$ respectively, so that $OM = \cos u$, $MP = \sin u$, $ON = \cosh v$, $NQ = \sinh v$.



On the unit-circle take, in the negative sense, the arc $JP' = IP$ and let OP' meet the concentric circle, rad. ON or $\cosh v$, in Q': then $P'M'$, $Q'N'$ being ordinates to OI. $OM' = \sin u$.

$$ON' = \cosh v \,.\, \sin u = x,\ M'P' = \cos u,\ N'Q' = \cosh v \,.\, \cos u,$$

and if Z is taken in $N'Q'$ such that

$$N'Z : N'Q' :: \sinh v : \cosh v,\ N'Z = \sinh v \,.\, \cos u = y\,;$$

hence

$$\overline{OZ} = \cosh v \,.\, \sin u + i \sinh v \cos u = \sin (u + vi).$$

Now if the ordinates of a circle to a given diameter are reduced in a given ratio, they become the ordinates of an ellipse of which that diameter is the major axis. Hence if V is taken in OJ such that $OV = \sinh v$, Z is a point in the ellipse whose semi-axes are ON, OV, and since $ON^2 - OV^2 = \cosh^2 v - \sinh^2 v = 1$, the foci of the ellipse are I, I' and its excentricity $= OI/ON = \operatorname{sech} v$.

Again since $\quad ON' = \cosh v \,.\, \sin u = ON \sin u,$

and $\quad N'Z = \sinh v \,.\, \cos u = NQ \cos u,$

for a given value of u, ON', $N'Z$ are respectively the abscissa ON and the ordinate NQ of the excircle changed in fixed ratios, and therefore Z is a point in the hyperbola, with the focal semi-axis OM' and the conjugate semi-axis equal to $M'P'$, of which $OP'Q'$ is an asymptote. Also since

$$OM'^2 + M'P'^2 = OI^2,$$

I, I' are the foci of the hyperbola.

Hence Z is determined as the intersection of an ellipse and hyperbola of the system of confocal conics, whose foci are I and I'. It is a well-known property of a system of confocal conics, that one ellipse and one hyperbola of the system may be drawn through any point in the plane and that these curves cut one another orthogonally. Hence $\overline{OZ}$ or $\sin (u + vi)$ may have any value whatever of the form $x + yi$.

If v is constant and u increases continuously from zero so that P moves round the unit circle from I in the positive direction, or that from I through J to I', Z moves on its ellipse from V in the contrary direction, or that through N to V'. Initially the hyperbola on which Z lies becomes the straight line (or double line) coincident with OJ: as u increases, the vertex of the hyperbola moves from O towards I, while its asymptotes close in towards OI, and when $u = 1^{\llcorner}$, the hyperbola becomes that part of the indefinite line through I', O, I which lies beyond I and I'.

In the case where $v = 0$, so that $\overline{OZ} = \sin u$, the ellipse degenerates into the finite line IOI', and the variations of $\sin u$, where u is scalar, are represented by oscillations along IOI' from O to I, then back through O to I', and thence forward to O, repeated over and over again for each complete circuit of the unit circle by P. The student will observe what a very small portion of the variations of the complete function $\sin(u + vi)$ correspond to those of the ordinary trigonometrical function $\sin u$, where u is scalar.

If u is constant and v increases continuously from zero, Z moves on a fixed hyperbola, and the ellipse determining its position starting from the finite line $I'OI$, which cuts the hyperbola in M', increases continually till its axes become infinite, when $v = \infty$.

It thus appears that $\sin(u + vi)$ is periodic with respect to changes of u, but non-periodic with respect to changes of v.

Since $\cos(u + vi) = \sin(1 + u + vi)$, the vector $\cos(u + vi)$ or $\overline{OZ_1}$ will be determined by the same series of confocal conics, only substituting $1 + u$ for u. This substitution determines a radius, OQ'_1, of the auxiliary circle, diameter NN', at right angles to OQ' and *behind* it relatively to the positive direction of rotation, and Q'_1 determines a point Z_1 on the ellipse such that OZ_1 is a semi-diameter conjugate to OZ. Also Z_1 lies on the hyperbola whose semi-axes are $\cos u$, $\sin u$ respectively, and whose asymptotes are therefore at right angles to those of the hyperbola on which Z lies.

9. *cosh* $(u + vi)$ and *sinh* $(u + vi)$.

We have

$$\cosh(u + vi) = \cos(iu + vi^2) = \cos(-v + iu) = \cos(v - iu),$$

and

$$\sinh(u+vi) = -i\sin(iu+vi^2) = i\sin(v-iu).$$

Hence the vector $\cosh(u+vi)$ is the vector determined by the intersection of the ellipse, whose semi-axes are $\cosh u$, $\sinh u$, and the hyperbola, whose semi-axes are $\cos v$, $\sin v$, which are an ellipse and hyperbola of the confocal system, foci I, I'. The vector $\sinh(u+vi)$ however is determined by the intersection of the ellipse, whose semi-axes are $\sinh u$, $\cosh u$ and whose foci are therefore, not I, I', but J, J', with the hyperbola conjugate to the former, whose semi-axes are $\sin v$, $\cos v$, and of which the foci are therefore also J, J'.

Thus, while $\sin(u+vi)$, $\cos(u+vi)$ are conjugate semi-diameters of the same ellipse, where it is cut by two hyperbolas of the same confocal system, whose semi-axes are equal, the focal axis of the one to the non-focal of the other; $\sinh(u+vi)$, $\cosh(u+vi)$ are conjugate semi-diameters of an hyperbola and its conjugate, where they are cut, the one by an ellipse of the same system, foci I, I', and the conjugate hyperbola by an equal ellipse of the conjugate system, foci J, J'.

10. *tan* $(u+vi)$.

Let

$$z = x+yi = \tan(u+vi);$$

then, leaving the details of the calculation for verification by the student,

$$x = \frac{\sin 2u}{\cos 2u + \cosh 2v}, \quad y = \frac{\sinh 2v}{\cos 2u + \cosh 2v},$$

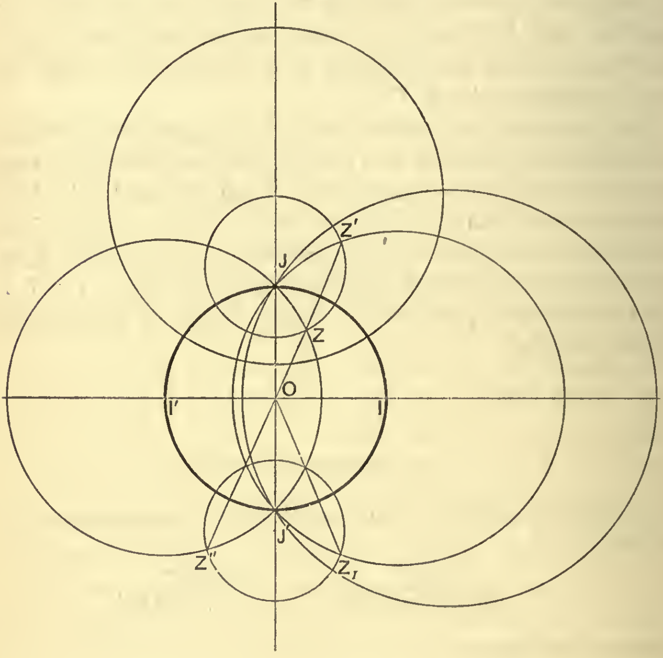
whence, eliminating u,

$$x^2 + y^2 - 2y\coth 2v + 1 = 0,$$

and eliminating v,

$$x^2 + y^2 + 2x\cot 2u - 1 = 0$$

The former of these two equations represents a series of non-intersecting co-axal circles, whose radical axis coincides with the prime vector OI, and whose centres lie on the line JOJ', the distance from O to the centre of any particular circle being $\coth 2v$ and its radius $\operatorname{cosech} 2v$. The latter represents a series of circles all passing through J, J', whose centres therefore lie on the line IOI', the distance from O to the centre of any particular circle being $-\cot 2u$ and its radius $\operatorname{cosec} 2u$. These two series of coaxal circles, as is well known, cut each other orthogonally.



Hence the vector $\tan(u+vi)$ is represented by $\overline{OZ}$, where Z is one of the points of intersection of one of the first system of circles, determined by the value of v, with one of the other system, determined by the value of u. The signs of the expressions for x, y will determine which of the two points is to be taken for the given values of u and v.

11. *cot* $(u+vi)$.

From the construction for $\tan(u+vi)$ that for $\cot(u+vi)$ may be obtained either by the relation $\cot(u+vi)=\dfrac{1}{\tan(u+vi)}$ or by $\cot(u+vi)=\tan(\overline{1-u-vi})$. It will be found that if in the diagram $\overline{OZ}=\tan(u+vi)$ and if OZ is produced to meet either of the circles which determine Z, the one in Z' or the other in Z'', then $\overline{OZ}_1$, which is symmetrical with $\overline{OZ'}$ with respect to the axis IOI' or with $\overline{OZ''}$ with respect to JOJ', represents $\cot(u+vi)$.

12. *tanh* $(u+vi)$ *and coth* $(u+vi)$.

Again we have

$$\tanh(u+vi)=-i\tan(i\,.\,\overline{u+vi})=i\tan(v-ui)$$

$$\coth(u+vi)=i\cot(i\,.\,\overline{u+vi})=i\cot(-v+ui)=-i\cot(v-ui),$$

from which it appears that the values of $\tanh(u+vi)$ are those of $\tan(v-ui)$ turned through a right angle in the positive sense, while the values of $\coth(u+vi)$ are those of $\cot(v-ui)$ turned through a right angle in the negative sense. The common axis of the system of intersecting circles will then be IOI', and I, I' will be the limiting points of the non-intersecting system.

MISCELLANEOUS EXAMPLES.

(CHAPTERS I. TO VI.)

1. Prove that, if A, B, C denote the angles of a triangle,

(1) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

(2) $1 + \cos 2A + \cos 2B + \cos 2C = -4 \cos A \cos B \cos C.$

(3) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$

(4) The orthocentre H is the mean point of A, B, C with the weights $\tan A$, $\tan B$, $\tan C$ respectively.

(5) The circumcentre O is the mean point of A, B, C with the weights $\sin 2A$, $\sin 2B$, $\sin 2C$ respectively.

(6) Hence that, G being the mean point of A, B, C with equal weights, $\overline{HG} = 2 \,.\, \overline{GO}.$

2. Reduce $\frac{(x-b)(x-c)}{(a-b)(a-c)}$ to the form ri^u, when for x, a, b, c are substituted i^{2z}, i^{2u}, i^{2v}, i^{2w} respectively: and thence prove that, (S denoting the sum of symmetrical terms)

$$S\left\{\frac{\sin(z-v)\sin(z-w)}{\sin(u-v)\sin(u-w)} \cdot \genfrac{}{}{0pt}{}{\cos}{\sin}\,(2\,\overline{z-u})\right\} = 1$$

and that sinh and cosh may be substituted for sin and cos respectively in this formula.

3. $S\left\{\frac{\sin(z-v)\sin(z-w)}{\sin(u-v)\sin(u-w)}\right\} = 1.$

4. $S\left\{\frac{\sin(z-v)\sin(z-w)}{\sin(u-v)\sin(u-w)} \cdot \genfrac{}{}{0pt}{}{\cos}{\sin}\,(2u)\right\} = \genfrac{}{}{0pt}{}{\cos}{\sin}\,(2z).$

5. If $i^u + i^v = m + ni$, prove that $\sin(u+v) = \frac{2mn}{m^2+n^2}$,

$$\cos(u-v) = \frac{m^2+n^2}{2} - 1, \quad \cos u \cos v = \frac{m^2+n^2}{4} - \frac{n^2}{m^2+n^2},$$

and

$$\cos 2u + \cos 2v = \frac{m^2-n^2}{m^2+n^2}(m^2+n^2-2).$$

6. If $\cosh u + \cosh v = m$ and $\sinh u + \sinh v = n$,

$$\sinh(u+v) = \frac{2mn}{m^2-n^2},\ \cosh(u-v) = \frac{m^2-n^2}{2} - 1,$$

and
$$\cosh 2u + \cosh 2v = \frac{m^2+n^2}{m^2-n^2}(m^2-n^2-2).$$

7. $\tan(u+vi) = \dfrac{\sin 2u + i\sinh 2v}{\cos 2u + \cosh 2v}$

and
$$\tanh(u+vi) = \frac{\sinh 2u + i\sin 2v}{\cosh 2u + \cos 2v}.$$

8. Prove that

$$(1+i\sinh v)^{\frac{1}{2}} + (1-i\sinh v)^{\frac{1}{2}} = 2\cosh\left\{\lambda i + (-1)^{\lambda}\frac{v}{2}\right\},$$

where λ is any integer: and that the expression has the four values

$$\pm 2\cosh\frac{v}{2},\ \pm 2\sinh\frac{v}{2}.$$

Verify this by putting iv for v.

9. Prove that, if u and $u+h$ are positive acute angles, $\sin(u+h)/\sin u$ is less, and $\tan(u+h)/\tan u$ greater than $1 + \dfrac{h}{u}$: and that, as u increases from 0 to $1^{\llcorner}$, $\sin u/u$ decreases continuously from $\dfrac{\pi}{2}$ to 1 or from 1 to $\dfrac{2}{\pi}$, according as u is measured in right angles or in radians, while $\tan u/u$ increases continuously from $\dfrac{\pi}{2}$ to ∞ or from 1 to ∞.

10. If $\sin v = m\sin u$, where m is a constant less than 1, as u increases from 0 to $1^{\llcorner}$, $\tan v/\tan u$ decreases continuously from m to 0 and $\sin(u-v)/\sin(u+v)$ increases continuously from $\dfrac{1-m}{1+m}$ to 1.

11 Prove that $\left(\cos\dfrac{\theta}{n}\right)^m$ lies between 1 and $1 - \dfrac{m\theta^2}{2n^2}$, m being positive and θ the measure of an angle in radians: and thence that the limit of $\left(\cos\dfrac{\theta}{n}\right)^n$, when n becomes infinite, is 1. Also the limit of $\left(\cos\dfrac{\theta}{n}\right)^{n^2} = e^{-\frac{\theta^2}{2}}$.

12. The complete value of i^i is a series of scalars in geometrical progression. Does it follow from $a^i = b^i$ that $a = b$?

13. The prime value of $(1+i)^{1+i} . (1-i)^{1-i} = {\cdot}410\ldots$ and that of $(1+i)^{1+i}/(1-i)^{1-i} = i^{\log_\eta 2\eta}$.

14. Prove that $i^i + i^{-i} = 2\cosh(1)$ and $i^i - i^{-i} = -2\sinh(1)$.

15. Prove that $i^{m+ni} + i^{-m-ni} = \sqrt{2(\cos 2m + \cosh 2n)}\, i^v$, where $\tan v = \tan m . \tanh n$.

16. Prove that

$$\lambda_\eta (m+ni) = \tfrac{1}{2} \log_\eta (m^2+n^2) + i \tan^{-1}\frac{n}{m} \text{ and } \lambda_\eta \frac{m+ni}{m-ni} = 2i \tan^{-1}\frac{n}{m}.$$

In these formulae rectangular measure is implied, adapt them to the case of circular measure. Show that the two sides of the equations have the same multiplicity of values.

17. Prove that $\{\cos(u+vi) + i\sin(u+vi)\}^{m+ni} = \eta^{-(mv+nu)} i^{mu-nv}$.

18. $\{\cos(u+vi)\}^{m+ni} = \eta^{m\log_\eta \rho + nw} . i^{n\log_\eta \rho - mw}$, where

$$\rho = \sqrt{\frac{1}{2}(\cos 2u + \cosh 2v)} \text{ and } \tan w = \tan u . \tanh v.$$

19. $\cosh(u+vi) + i\sinh(u+vi) = \sqrt{\cosh 2u - \sin 2v} . i^w$

$$\text{where } \tan w = \tanh u \tan(v + \tfrac{1}{2}).$$

20. If u, v are the values of the sectors IOP, IOQ of the excircle, and the chord PQ meets the asymptotes in R, R', prove that

(1) $\overline{OR} = \dfrac{\eta^u + \eta^v}{\sqrt{2}} . i^{\frac{1}{2}}$, $\overline{OR'} = \dfrac{\eta^{-u} + \eta^{-v}}{\sqrt{2}} i^{-\frac{1}{2}}$.

(2) $\overline{PR} = -\overline{QR'} = \dfrac{\eta^v}{\sqrt{2}} i^{\frac{1}{2}} - \dfrac{\eta^{-u}}{\sqrt{2}} i^{-\frac{1}{2}}$.

(3) $\overline{QR} = -\overline{PR'} = \dfrac{\eta^u}{\sqrt{2}} i^{\frac{1}{2}} - \dfrac{\eta^{-v}}{\sqrt{2}} i^{-\frac{1}{2}}$.

(4) $PR . PR' = QR . QR' = \cosh(u+v) =$ const. for all chords parallel to PQ.

(5) $OR . OR' = 2\cosh^2 \dfrac{u-v}{2} =$ const. if the area of the sector POQ is constant.

21. Show that, if a line cuts off lengths h, k from the asymptotes, the vector radii of its intersections with the excircle are

$$\tfrac{1}{2}(hi^{\frac{1}{2}} + ki^{-\frac{1}{2}}) \pm \left(1 - \frac{2}{hk}\right)^{\frac{1}{2}} . \tfrac{1}{2}(hi^{\frac{1}{2}} - ki^{-\frac{1}{2}}).$$

Examine the cases where h, k are of contrary signs, and deduce the properties of conjugate diameters.

22. Prove that (see fig. p. 134) Q, Q' being the points $u + v$, $u -$ respectively

$$\overline{OQ} . \overline{OQ'} = \cosh 2v + i \sinh 2u = \overline{OP^2} + 2 \sinh^2 v,$$

and $$OQ . OQ' = \{\cosh 2\overline{u + v} . \cosh 2\overline{u - v}\}^{\frac{1}{2}}.$$

23. Prove that $\overline{R'Q} . \overline{QR} = \overline{PL^2}$ and $R'Q . QR = OP^2$.

24. Prove that $\overline{OR} . \overline{OR'} = OR . OR' = \cosh^2 v$.

25. The tangent at P passes through T.

26. If $\overline{OP} = \cos(u + vi)$, $\cosh v - \cos u = IP$, $\cosh v + \cos u = I'P$,
$\cosh 2v + \cos 2u = 2OP^2$, $\cosh 2v - \cos 2u = 2IP . I'P$.

27. If $\overline{OP} = x + yi$,

(1) $\sin^{-1}(x + yi) = \sin^{-1}\dfrac{I'P - IP}{2} + i \cosh^{-1}\dfrac{I'P + IP}{2}$.

(2) $\cos^{-1}(x + yi) = \cos^{-1}\dfrac{I'P - IP}{2} + i \cosh^{-1}\dfrac{I'P + IP}{2}$.

(3) $\tan^{-1}(x + yi) = \tfrac{1}{2}(PJJ' + PJ'J) + i \log_\eta \dfrac{J'P}{JP}$.

28. Prove the identities :

$$S\left\{x^2\frac{(x+y)\,(x+z)}{(x-y)\,(x-z)}\right\} = (x + y + z)^2$$

$$S\{\sin 4u \cot (w - u). \cot (u - v)\}$$
$$= 2 (\cos 2u + \cos 2v + \cos 2w) (\sin 2u + \sin 2v + \sin 2w).$$

$$S\{\cos 4u \cot (w - u) \cot (u - v)$$
$$= 2\{\cos (2u + \tfrac{1}{2}) + \cos (2v + \tfrac{1}{2}) + \cos (2w + \tfrac{1}{2})\} \{\sin (2u + \tfrac{1}{2}) + \sin (2v + \tfrac{1}{2}) + \sin (w + \tfrac{1}{2}\}.$$

CHAPTER VII

ROOTS OF UNITY

1. We have seen (c. ii. §§ 10 and 11) that the n^{th} root or $1/n^{\text{th}}$ power of any (scalar or) vector has n and only n distinct values; the simplest of which, or that which is first reached in revolving in the positive sense from the prime vector, we have termed its *prime* value, (this being in the case of a positive scalar its *arithmetical* value); and that the other values are obtained by multiplying the prime value by the n values of the n^{th} root of the prime vector or 1. These quantities, the n^{th} roots of unity, have many remarkable properties, the simplest and most important of which we proceed to discuss in the present chapter.

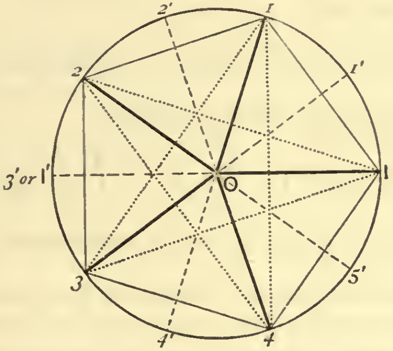
2. *Geometrical representation of* $1^{\frac{1}{n}}$ and $(-1)^{\frac{1}{n}}$.

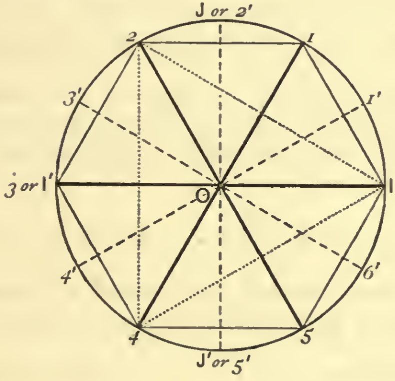
Since, when λ is any integer,

$$\overline{OI} \text{ or } 1 = i^{4\lambda} = \cos 4\lambda + i \sin 4\lambda,$$

$$\overline{OI}^{\frac{1}{n}} \text{ or } 1^{\frac{1}{n}} = i^{\frac{4\lambda}{n}} = \cos \frac{4\lambda}{n} + i \sin \frac{4\lambda}{n},$$

so that any value of $1^{\frac{1}{n}}$ is a *unit vector* inclined to the prime vector at the angle $4\lambda/n$, that is, some multiple of one n^{th} part of 4 right angles. Hence, if the circumference of the unit circle is divided into n equal arcs at points denoted by I, 1, 2, 3 . . . . $(n-1)$, the radii drawn to these points are the several values of the n^{th} roots of unity, and it is thus plain that there are n, and only n, distinct vectors representing the values of $1^{\frac{1}{n}}$. Of these values one only, $(\overline{OI})$ is scalar if n is an odd number, and two only, $(\overline{OI}$ and $\overline{OI'})$ if n is even. The diagrams, drawn for the cases, $n = 5$ and $n = 6$, make this evident. It should be noted that, when n is even, the extremities of the vectors form a system of points symmetrical with respect to both IOI' and JOJ', while, when n is odd, the system is symmetrical with respect to IOI' and not with respect to JOJ': in the former case too the





system is symmetrical with respect to the centre O, the points being the opposite extremities of a set of diameters, but not so in the latter.

The values of $(-1)^{\frac{1}{n}}$ might in like manner be deduced from the consideration that

$$-1 = i^{4\lambda+2} = \cos(4\lambda + 2) + i \sin(4\lambda + 2);$$

but we may also deduce them as follows.

The *prime* value of $(-1)^{\frac{1}{n}} = i^{\frac{2}{n}}$, and the general value is $i^{\frac{2}{n}} . i^{\frac{4\lambda}{n}}$, so that the system of vectors corresponding to $(-1)^{\frac{1}{n}}$ is simply that corresponding to $1^{\frac{1}{n}}$ advanced through the angle $2/n^{\llcorner}$, so that each vector of the former bisects the angle between two consecutive vectors of the latter. These vectors are represented in the diagram above by the broken lines drawn to the points $1'$, $2' \ldots n'$, and it is obvious that, if n is even, none of the values are scalar, while, if n is odd, one only (OI') is scalar.

The n^{th} roots of 1 are the roots of the binomial equation $z^n - 1 = 0$, so that if n is odd, omitting the factor $z - 1$ corresponding to the arithmetical root 1, the non-scalar roots are the roots of the equation

$$z^{n-1} + z^{n-2} + z^{n-3} + \ldots + z^2 + z + 1 = 0:$$

while if n is even, omitting the factor $x^2 - 1$ corresponding to the scalar roots ± 1, the non-scalar roots are those of

$$z^{n-2} + z^{n-4} + \ldots + z^4 + z^2 + 1 = 0.$$

So also the n^{th} roots of -1 are the roots of the binomial equation $z^n + 1 = 0$, so that, if n is odd, omitting the factor $z + 1$ corresponding to the scalar root -1, the non-scalar roots are those of the equation

$$z^{n-1} - z^{n-2} + \ldots + z^2 - z + 1 = 0,$$

while, if n is even, there are no scalar roots.

Ex.—What are the roots of $z^{n-1} - z^{n-2} + \ldots - z^2 + z - 1 = 0$, n being an even number?

3. *The Roots as Powers of the Principal Root.*

The general expression for the n^{th} roots of unity is $i^{\frac{4\lambda}{n}}$, an expression which has (as we have seen) n distinct values for any n consecutive values of λ, repeated in cycles of the same n values in the same sequence, as the values of λ are continued upwards or downwards beyond the limits of the first set. These values are the consecutive powers of $i^{\frac{4}{n}}$, which may therefore be conveniently called the *principal* n^{th} root. Denoting the principal root by α, we have $\alpha = i^{\frac{4}{n}} = \cos\frac{4}{n} + i\sin\frac{4}{n}$,

represented by the vector $\overline{O1}$: and the n roots are

$$1,\ \alpha,\ \alpha^2 \ldots,\ \alpha_{n-1}$$

corresponding to the vectors $\overline{OI}, \overline{O1}, \overline{O2} \ldots \overline{O(n-1)}$.

Since $\alpha^{n-p} = \alpha^n.\alpha^{-p} = \alpha^{-p}$, the roots may often be conveniently grouped in pairs, corresponding to points symmetrically situated on opposite sides of the prime vector OI, thus:

$$1,\ \begin{matrix} \alpha, & \alpha^2, & \ldots, & \alpha^{\frac{n-1}{2}} \\ \alpha^{-1}, & \alpha^{-2} & \ldots, & \alpha^{-\frac{n-1}{2}} \end{matrix} \qquad \text{when } n \text{ is odd}$$

or

$$1,\ \begin{matrix} \alpha, & \alpha^2 & \ldots, & \alpha^{\frac{n}{2}-1} \\ \alpha^{-1}, & \alpha^{-2} & \ldots, & \alpha^{-\left(\frac{n}{2}-1\right)} \end{matrix} \ -1 \text{ when } n \text{ is even}$$

It is to be observed for any pair α^p, α^{-p}, that

$$\alpha^p + \alpha^{-p} = i^{\frac{4p}{n}} + i^{-\frac{4p}{n}} = 2\cos\frac{4p}{n}$$

$$\text{and } a^p - a^{-p} = i^{\frac{4p}{n}} - i^{-\frac{4p}{n}} = 2i \sin \frac{4p}{n},$$

as is evident from the diagram.

4. *Primary and Subordinate Roots.*

If n is a prime number, the successive powers of *any one* of the roots (except 1) will determine all the roots in a special sequence. For, taking the root a^p, $(a^p)^q$ and $(a^p)^{q'}$ can only be equal if $q = q'$ or pq and pq' differ by a multiple of n, in which case, p being prime to n, q and q' must differ by n or a multiple of n, and this for values of q, q' less than n is impossible. Hence, giving to q the successive values $0, 1, 2 \ldots, (n-1)$, we shall have all the roots in the sequence $1, a^p, a^{2p}, \ldots, a^{(n-1)p}$; a sequence determined by the successive remainders of $p, 2p, 3p \ldots (n-1)p$ when divided by n.

Thus in the case of $n=5$, the sequence for a being 0, 1, 2, 3, 4, that for a^2 is 0, 2, 4, 1, 3: that for a^3 is 0, 3, 1, 4, 2, or that for a^2 reversed in order: that for a^4 is 0, 4, 3, 2, 1, or that for a reversed in order. If the successive points on the unit circle are joined in accordance with these sequences as in Fig. 1, a determines the regular convex pentagon I 1 2 3 4, and a^2 the regular crossed pentagon I 2 4 1 3, while a^4 (or a^{-1}) and a^3 or (a^{-2}) determine the same respectively traversed in the opposite sense.

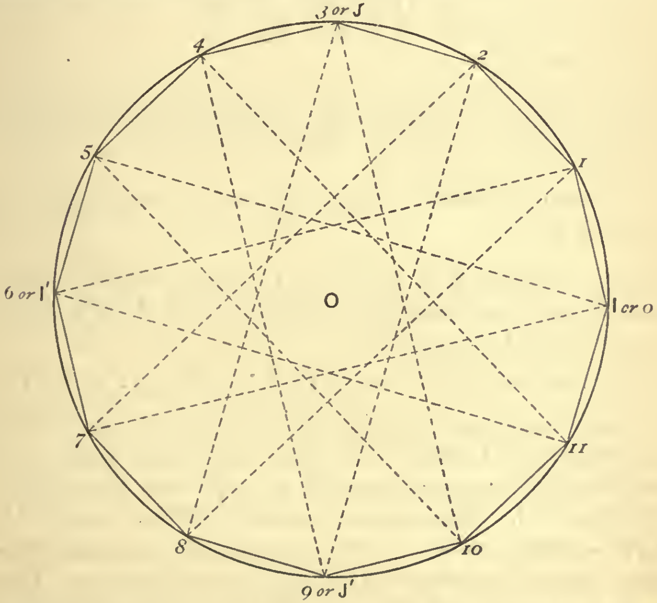
If n is a composite number, the same reasoning holds good if p is prime to n. But if p is not prime to n, and p/n reduced to its lowest terms becomes p'/n', $(a^p)^q$ and $(a^p)^{q'}$ will be equal if $p'q$ and $p'q'$ differ by a multiple of n', that is, if q and q' differ by n' or a multiple of n'. In this case the successive powers of the root a^p determine only n' out of the n roots. The complete period determined by the first n powers $(0, 1, 2 \ldots \overline{n-1})$ splitting up into subordinate periods of n' roots, repeated as many times as n contains n'.

Those roots, whose successive powers determine *all* the n roots, may be termed *primary*, while the other roots, which determine only a certain set out of the n roots, (in number a submultiple of n) may be termed *subordinate*.

Then if n is a prime number, all the roots (except 1) are primary roots; but if n is a composite number, there are only as many primary roots as there are numbers less than n and prime to it, and this number is, if n_1, n_2, n_3 . . are the prime factors of n, (by a well-known formula)* $n(1-\frac{1}{n_1})(1-\frac{1}{n_2})(1-\frac{1}{n_3})\ldots$ or, $n_1^{k_1-1}\, n_2^{k_2-1}\ldots(n_1-1)(n_2-1)\ldots$. if $n = n_1^{k_1}.\, n_2^{k_2}\ldots$

If p is a divisor of n so that $n = pm$, any p^{th} root is also an n^{th} root: for if $\beta^p = 1$, $\beta^n = (\beta^p)^m = 1^m = 1$. Hence among the secondary n^{th} roots of 1 will be found all the p^{th} roots.

No primary root of one order can be also a primary root of another order. For any primary root of the n^{th} order is of the form $i^{\frac{4p}{n}}$, where p is prime to n, and p/n cannot be equal to p'/n', where p' is prime to n', unless $p = p'$ and $n = n'$.



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* See C. Smith's *Alg.*, Art. 380, Ch. xxviii., or Hall and Knight, Art. 431, Ch. xxx.

As an instance, let $n = 12 = 2^2 \times 3$.

The principal root is $\alpha = i^{\frac{4}{12}} = i^{\frac{1}{3}} = \cos\frac{1}{3} + i\sin\frac{1}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

There are $2^1 \times 3^0 (2-1).(3-1)$, or 4, primary 12th roots, and these are $\alpha,\ \alpha^5,\ \alpha^7,\ \alpha^{11}$ or $\frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2}i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}i$.

Joining the extremities of the successive powers of each vector in order, we have the regular dodecagon and the regular crossed dodecagon, each taken in the two opposite senses, as shown in the diagram.

The secondary roots are $\alpha^2, \alpha^3, \alpha^4, \alpha^6, \alpha^8, \alpha^9, \alpha^{10}$; of which $\alpha^2, \alpha^4, \alpha^6, \alpha^8, \alpha^{10}$ are the 6th roots $\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -1,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$;

$\alpha^3, \alpha^6, \alpha^9$ are the fourth roots $i,\ -1,\ -i$;

α^4, α^8, are the cube roots $-\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i$;

α^6 is the square root -1.

Joining the points determined by these sequences, the first gives (Fig. 2, § 2) the regular hexagon, the second the square $IJI'J'$, the third the equilateral triangle $I\,24$, and the last the diameter II'.

5. *Roots of any order determined by roots of orders, which are powers of primes.*

Let p be prime to q, and let α, β be the principal pth and qth roots of unity respectively, then the pqth roots of unity are the several terms in the product

$$(1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1})(1 + \beta + \beta^2 + \ldots + \beta^{q-1}).$$

To prove this, observe that the terms in each factor are all different and, except 1, there is no term the same in both. Next every term in the product, as $\alpha^r\beta^s$, is a pqth root, for $(\alpha^r\beta^s)^{pq} = (\alpha^p)^{qr}(\beta^q)^{ps} = 1$. Also no two terms $\alpha^r\beta^s$, $\alpha^{r'}\beta^{s'}$ can be equal; for, if $\alpha^r\beta^s = \alpha^{r'}\beta^{s'}$, then $\alpha^{r-r'} = \beta^{s'-s}$, that is, a pth root equal to a qth root. Lastly, the number of terms in the product is pq. Hence the pq terms, being all different, and each a pqth root of 1, these terms are all the pqth roots.

Let r be prime to both p, q and therefore prime to pq, and γ the principal r^{th} root. Then the terms in the product of the sum of the pq^{th} roots by that of the r^{th} roots are, for a like reason, the pqr^{th} roots: so that the pqr terms in the product

$$(1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1})(1 + \beta + \beta^2 + \ldots + \beta^{q-1})(1 + \gamma + \gamma^2 + \ldots + \gamma^{r-1})$$

are the pqr^{th} roots of unity.

It is obvious that this reasoning may be extended to any number of factors $p, q, r, s \ldots$, each of which is prime to the rest.

If n is a composite number, whose prime factors are $n_1, n_2, n_3 \ldots$ and $n = n_1^{k_1} . n_2^{k_2} . n_3^{k_3} \ldots$, then $n_1^{k_1}, n_2^{k_2}, \ldots$ are prime to one another, and therefore, by what we have just proved, the n^{th} roots of unity are all the products that can be formed from the $n_1^{k_1}$th, $n_2^{k_2}$th . . .roots.

Thus since $12 = 2^2 \times 3$, the products of the 3 cube roots and the 4 fourth roots give the 12 twelfth roots.

Again since $360 = 2^3 \times 3^3 \times 5$, all the 360th roots may be obtained as the products of one of the eighth roots, one of the ninth roots and one of the fifth roots.

The multiplication may be performed mechanically thus. Take three equal circles divided the first into 5, the second into 8, and the third into 9 equal arcs. Place the second on the first so that the circles coincide, the zero of the second being made successively to coincide with 0, 1, 2, 3, 4 of the first, and in each position mark on the first the points coincident with the eight points of the second; then the points on the first (none of which will be coincident) will determine the 40 fortieth roots of 1. Do the same with the third, placing its zero successively on these 40 points, and there will then be 360 points on the first corresponding to the 360th roots of unity.

This process is obviously quite general, and might be applied to demonstrate the general case discussed in this article.

7. *The sum of the* n^{th} *roots* $= 0$.

The points on the unit circle corresponding to the n roots are symmetrically placed with respect to OI, and also with respect to

$O1$, $O2$, . . : hence their mean point must lie on each of these lines and must therefore be O itself, so that

$$\overline{OI} + \overline{O1} + \overline{O2} + \dots + \overline{O(n-1)} = 0$$

or
$$1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0.$$

This may also be proved algebraically thus.

Let $S = 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1}$;

then $\alpha S = \alpha + \alpha^2 + \dots + \alpha^{n-1} + \alpha^n = S$, since $\alpha^n = 1$

whence, α not being 1, $S = 0$.

Substituting for α the trigonometrical form $\cos\frac{4}{n} + i\sin\frac{4}{n}$,

we have both $1 + \cos\frac{4}{n} + \cos\frac{8}{n} + \dots + \cos\frac{4(n-1)}{n} = 0$

and
$$\sin\frac{4}{n} + \sin\frac{8}{n} + \dots + \sin\frac{4(n-1)}{n} = 0.$$

The truth of this latter series is obvious, since the terms cancel in pairs, the first with the last, the second with the last but one, and so on.

The same is true of the first series, if n is even, the terms in the first half being equal and of contrary signs to the terms in the second half: but, if n is odd, the terms do not cancel in pairs. All this is evident from the geometrical representation.

Ex.—By the first series prove that $\cos\frac{2}{5} = \frac{\sqrt{5}+1}{4}$.

8. *The sum of the* m^{th} *powers of the* n^{th} *roots* $= 0$ *or* n, *as* m *is not, or is, a multiple of* n.

Let $S_m = 1 + \alpha^m + (\alpha^2)^m + \dots + (\alpha^{n-1})^m$

then $\alpha^m S_m = \alpha^m + (\alpha^2)^m + (\alpha^3)^m + \dots + (\alpha^{n-1})^m + (\alpha^n)^m = S_m$, since $\alpha^n = 1$ therefore $S_m = 0$, unless $\alpha^m = 1$, or $m = n$ or a multiple of n. If $m = n$ or a multiple of n, α^m, α^{2m}, . . . each $= 1$, so that S_n, S_{2n}, S_{3n}, . . . each $= n$.

Otherwise, $S_m = 1 + a^m + (a^m)^2 + (a^m)^3 + \ldots + (a^m)^{n-1}$, whence, if a^m is a *primary* root, the n terms of S_m give the n n^{th} roots, whose sum (by § 7) is 0. If a^m is a secondary root, m not being prime to n, let $m = km'$ and $n = kn'$, then a^k is an n'^{th} root, and therefore $1 + a^k + a^{2k} + \ldots + a^{(n'-1)k} = 0$ and S_m splits into k subordinate periods of n' terms, each separately $= 0$, and therefore $S_m = 0$.

As in the last article, the following trigonometrical series result:

$$1 + \cos\frac{4m}{n} + \cos\frac{8m}{n} + \ldots + \cos\frac{4(n-1)m}{n} = 0 \quad \text{generally,}$$

but $= n$, if $m = kn$;

$$\sin\frac{4m}{n} + \sin\frac{8m}{n} + \ldots + \sin\frac{4(n-1)m}{n} = 0 \text{ always.}$$

9. *The sum of the products of every r of the n^{th} roots $= 0$, r being less than n.*

If β is any n^{th} root of unit, $x^n - 1 = x^n - \beta^n$, so that $x^n - 1$ is divisible by $x - \beta$: hence, a denoting the principal n^{th} root, $x^n - 1 = (x-1)(x-a)(x-a^2) \ldots (x - a^{n-1})$.

Fuller development of the consequences of this resolution into factors is reserved for a later chapter (Ch. X.). Here it is only necessary to observe that, since the coefficients of $x^{n-1}, x^{n-2}, \ldots x^2, x$ all vanish in the product of the factors above, and these coefficients are respectively the sums of all the products of the roots taken $1, 2, 3 \ldots \overline{n-1}$ together, these sums must all vanish.

If the product of all the n roots is taken, that product, which is equal to $a^{\frac{n(n-1)}{2}}$, is $+1$ or -1, as n is odd or even.

Further, since $(x-a)(x-a^2) \ldots (x-a^{n-1}) = \dfrac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots + x + 1$, the sum of the products of the roots, *excluding* 1, are $+1$ or -1, according as they are products of an even number or of an odd number of roots.

These results may be seen to be true from the following general considerations.

Every product of a number of roots is itself a root. For the product of any two or more vectors of the series $\overline{OI}$, $\overline{O1}$, . . . $\overline{O(n-1)}$ gives another vector of the same series.

For every term of the product a^m, there will be another term a^{-m}. For instance take the term $a^p\ a^q\ a^r$, then the term $a^{n-p}\ a^{n-q}\ a^{n-r}$ $= a^{-p}.\ a^{-q}.\ a^{-r}$ and these terms will be different and symmetrical with respect to OI, except in the case of $p + q + r = n$ or a multiple of n when the product will be unique and equal to 1; or, in the case of n even, when $p + q + r = \frac{n}{2}$ or an odd multiple of $\frac{n}{2}$, when the product will also be unique and equal to $-$ 1. Hence the series of roots obtained from the several products can be arranged in pairs symmetrical with respect to $\overline{OI}$ or 1. In like manner they can be arranged in pairs symmetrical with respect to $\overline{O1}$, or $\overline{O2}$, or $\overline{O3}$ . . , and therefore their sum must be 0. The single exception to this is the unique product of all the roots, which = 1 or $-$ 1, as n is odd or even.

10. *Application of the foregoing results.*

Having now established the principal elementary properties of the roots of unity, which find useful applications in various branches of analysis, especially in the Theory of Equations, we will conclude the chapter with a few simple instances of such use.

(1) Let α be the principal mth root, and β the principal nth root, of unity, m being prime to n; then the product of all possible factors of the form $a\alpha^r + b\beta^s$, where r may have any value from 0 to $m - 1$ inclusive, and s any value from 0 to $n - 1$ inclusive, will be a scalar function of a and b.

For, the values of α^r being the mth roots of unity, the sums of their products taken 1, 2, 3 . . . $m - 1$ together all vanish, and therefore in the product $(a + b)(a\alpha + b) \,.\,.\, (a\alpha^{m-1} + b)$ the only terms which have finite coefficients are those containing a^m and b^m, so that, since the product $\alpha . \alpha^2 \,.\,.\, \alpha^{m-1} = +1$, when m is odd and -1, when m is even,

$$(a + b)(a\alpha + b) \dots (a\alpha^{m-1} + b) = (-1)^{m-1}a^m + b^m.$$

For b, write successively $b\beta$, $b\beta^2 \dots b\beta^{n-1}$ in this identity, and multiply the results; then the product of the mn binomial factors of the form $a\alpha^r + b\beta^s$ is equal to

$$\left((-1)^{m-1}a^m + b^m\right)\left((-1)^{m-1}a^m + b^m\beta^m\right)\dots\left((-1)^{m-1}a^m + b^m\beta^{(n-1)m}\right).$$

Then since n is prime to m, 1, $\beta^m, \beta^{2m}, \dots \beta^{(n-1)m}$ are all n^{th} roots and are all different, so that they are the n n^{th} roots of unity. Hence, reasoning as before, the product last written

$$= (-1)^{n(m-1)}a^{mn} + (-1)^{m(n-1)}b^{mn}.$$

Since m is prime to n, m and n cannot both be even. If they are both odd mn is odd, and this result takes the form $a^{mn} + b^{mn}$; if m is odd and n even, the form is $a^{mn} - b^{mn}$, while if m is even and n odd, it is $b^{mn} - a^{mn}$. For instance, let $m = 2$, $n = 3$ and put $\sqrt{x}$ for a and $\sqrt[3]{y}$ for b, then ω denoting the principal cube root or $-\frac{1}{2} + \frac{\sqrt{3}}{2} . i$,

$$(\sqrt{x} + \sqrt[3]{y})(-\sqrt{x} + \sqrt[3]{y})(\sqrt{x} + \omega\sqrt[3]{y})(-\sqrt{x} + \omega\sqrt[3]{y})$$
$$(\sqrt{x} + \omega^2\sqrt[3]{y})(-\sqrt{x} + \omega^2\sqrt[3]{y}) = -x^3 + y^2,$$

whence a rationalising factor may be obtained for any one of these binomial factors or the product of any number of them.

(2) Let ω be the principal cube root of 1, so that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$: then

$$(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega) = x^2 + y^2 + z^2 - yz - zx - xy$$

and $(x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega) = x^3 + y^3 + z^3 - 3xyz.$

Hence $$x^3+y^3+z^3-3xyz=0$$

or, arranging according to powers of x,

$$x^3-3yz.x+(y^3+z^3)=0$$

if $$x=-y-z \text{ or } -y\omega-z\omega^2 \text{ or } -y\omega^2-z\omega.$$

Comparing this equation with the cubic equation in the form

$$x^3-qx+r=0$$

$$y^3+z^3=r \text{ and } 3yz=q, \text{ whence } y^3-z^3=\sqrt{r^2-4\left(\frac{q}{3}\right)^3}$$

so that $$y^3=\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}} \qquad z^3=\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}$$

and $$-y=\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}},\ -z=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}$$

whence the three roots of the cubic are expressed in accordance with the solution, known as Cardan's.

(3) Gauss's famous discovery of the possibility of dividing the circumference of a circle into n equal parts without assuming any postulate of construction beyond the usual conventions of Geometry, when n is a prime number of the form $2^\lambda+1$, depends on the theory of the roots of unity. The following outline of the argument will perhaps be of interest to the student. For full details see Peacock's *Algebra*, ch. xlv.

It is shown in the Theory of Numbers that, if n is a prime number, there are always values of m less than n, such that the remainders after division by n of m, m^2, m^3 . . m^{n-1} are all different and so form a sequence in a definite order, without repetition, of the numbers 1, 2, 3 . . , $n-1$; and that the number of such values is the same as that of the numbers less than n and prime to it.

Hence, if any one of these values of m is taken, the series

$$a^m, a^{m^2}, a^{m^3} \ldots, a^{m^{n-1}},$$

where a is the principal n^{th} root, determines all the n^{th} roots of unity, except unity itself, in a definite sequence; which has this property, that the substitution of any one of the series for a does not alter the sequence, but merely the point at which it begins.

Thus if $n=5$, the series a^2, a^{2^2}, a^{2^3}, a^{2^4} gives the roots in the order a^2, a^4, a^3, a; and in this putting a^2 for a, the sequence becomes a^4, a^8, a^6, a^2, or, since $a^5=1$, a^4, a^3, a, a^2; putting a^3 for a, it becomes a, a^2, a^4, a^3 and so on. The crossed quadrilateral in the figure exhibits these sequences, in passing round it in the sense indicated by the arrowheads.

Let $\rho_1, \rho_2, \ldots \rho_{n-1}$ denote the n^{th} roots of 1, (excluding 1 itself) in a sequence of this kind, and let ω be the principal $\overline{n-1}^{\text{th}}$ root of 1 so that $\omega, \omega^2, \ldots \omega^{n-2}$, and $\omega^{n-1} (= 1)$ are the several $\overline{n-1}^{\text{th}}$ roots. Then, taking

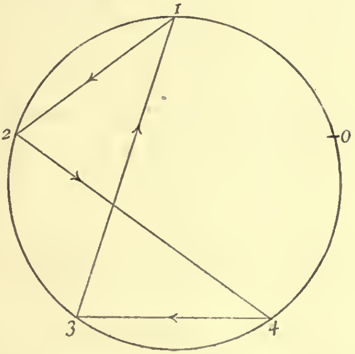
$$t = \rho_1 + \rho_2\omega + \rho_3\omega^2 + \ldots + \rho_{n-1}\omega^{n-2},$$

$$\omega t = \rho_{n-1} + \rho_1\omega + \rho_2\omega^2 \ldots + \rho_{n-3}\omega^{n-3} + \rho_{n-2}\omega^{n-2}$$

$$\omega^2 t = \rho_{n-2} + \rho_{n-1}\omega + \rho_1\omega^2 + \ldots + \rho_{n-4}\omega^{n-3} + \rho_{n-3}\omega^{n-2}$$

. . . .

$$\omega^{n-2} t = \rho_2 + \rho_3\omega + \rho_4\omega^2 + \ldots + \rho_{n-1}\omega^{n-3} + \rho_1\omega^{n-2}$$



But $t^{n-1} = (\omega t)^{n-1} = (\omega^2 t)^{n-1} = \ldots = (\omega^{n-2} t)^{n-1}$, since $\omega^{n-1} = 1$, therefore the expression for t^{n-1}, which may be reduced to the form $a_1 + a_2\omega + a_3\omega^2 + \ldots + a_{n-1}\omega^{n-2}$, is unchanged when the coefficients $\rho_1, \rho_2, \ldots$ are changed from term to term, the cyclical order remaining unchanged. Hence $a_1, a_2, \ldots a_{n-1}$ must be numerical quantities, not involving $\rho_1, \rho_2 \ldots$ explicitly.

In the case of $n = 5$, $t^4 = -1 + 4\omega + 14\omega^2 - 16\omega^3$. It will be a good exercise for the student to verify this by direct algebraical work, reducing the terms by means of the known properties of a and ω.

Thus t is one of the $\overline{n-1}^{\text{th}}$ roots of an integral function of ω with numerical coefficients, and if ω is known, it can be expressed as one of the $\overline{n-1}^{\text{th}}$ roots of an expression of the form $a + bi$. If $n - 1$ is of the form 2^p, ω can be found by repeated bisections of arcs of the unit circle, beginning with the semicircle, and the $\overline{n-1}^{\text{th}}$ root of $a + bi$ by repeated extractions of the square root (or geometrically finding successive mean proportionals) and repeated bisections of angles.

Thus if n is a prime number and of the form $2^p + 1$, the value of t can be found. The particular $\overline{n-1}^{\text{th}}$ root that must be taken must be one that is consistent with the equation, (found by putting $\omega = 1$) $t = \rho_1 + \rho_2 + . . + \rho_{n-1} = -1$ by the known property of the sum of the n^{th} roots. Let T_1 be the value of this root for the root ω, and T_2 T_3 . . T_{n-2}, its values when $\omega^2, \omega^3 . . \omega^{n-2}$ are substituted for ω, then we have the $\overline{n-1}$ equations

$$\rho_1 + \rho_2 + \rho_3 + . . + \rho_{n-1} = -1$$

$$\rho_1 + \rho_2\omega + \rho_3\omega^2 + = . + \rho_{n-1}\omega^{n-2} = T_1$$

$$\rho_1 + \rho_2\omega^2 + \rho_3\omega^4 + . . + \rho_{n-1}\omega^2 \qquad = T_2$$

. . . .

$$\rho_1 + \rho_2\omega^{n-2} + \rho_3\omega^{2n-4} . . + \rho_{n-1}\omega^{(n-2)(n-2)} = T_{n-2}$$

whence by addition, remembering that the sums of the powers of $\omega, \omega^2 . . . ., \omega^{n-2}$ vanish except when the powers are multiples of $n-1$,

$$(n-1)\,\rho_1 = -1 + T_1 + T_2 + . . + T_{n-2}$$

and by addition, after multiplying the successive equations by powers of ω as shown on the right hand side of the following, we have

$$(n-1)\;\rho_2 = -1 + T_1\omega^{n-2} + T_2\omega^{n-3} + . . . + T_{n-2}\omega$$

$$(n-1)\,\rho_3 = -1 + T_1\omega^{2n-4} + T_2\omega^{2n-6} + . . . + T_{n-2}\omega^2$$

&c.

Thus the $\overline{n-1}$ roots, different from 1, of the equation $x^n - 1 = 0$, where n is a prime number, are completely determined,* if the roots of $x^{n-1} - 1 = 0$ are known, and this is the case, as we have seen, when n is of the form $2^p + 1$.

The simplest case is that of $p = 2$ and therefore $n = 5$, and the detailed working out of this case will give a good indication of the laborious character of the investigation for other particular cases and of the great care that is necessary to obtain correctly the values of $T_1 T_2$, . . to be employed. The next case is that of $p = 4$ and therefore $n = 17$; this as well as the case of $n = 5$, is worked out in detail in Peacock's *Algebra* with some simplifications in the process, which, (the object of this summary being merely to show the general character of Gauss's investigation) it is needless here to discuss. The next value of p, which gives $2^p + 1$ as a prime number, is 8 and therefore $n = 257$.

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* By the *algebraical* solution of an equation with numerical coefficients is understood the expression of its roots exactly by means of definite arithmetical operations on finite numbers. This excludes the practical solution of the binomial equation $x^n - 1 = 0$ in the form $x = i^{\frac{4\lambda}{n}}$ or $\cos\frac{4\lambda}{n} + i\sin\frac{4\lambda}{n}$, from which, by the aid of a table of sines, approximate values of the roots can at once be obtained. The difficulty or impossibility of algebraical solutions in different cases is analogous to the difficulty or impossibility of geometrical constructions under the limitations imposed by the conventional postulates of construction in Geometry.

EXAMPLES.

1. The odd powers, from 1 to $2n-1$ inclusive, of the principal nth root of -1, determine all the nth roots, and their sum is zero. Also the sum of the even powers, from 0 to $2n-2$ inclusive, is zero.

2. The sum of the mth powers of the nth roots of -1 is zero, unless m is a multiple of n, when it is equal to $+n$ or $-n$, according as it is an even or an odd multiple.

3. Prove that $x^2+y^2+z^2-yz-zx-xy$ is unchanged, if x, y, z are each increased by the same quantity t.

4. If $U=x^3+y^3+z^3-3xyz$, and U becomes U', when $x+t$, $y+t$, $z+t$ are substituted for x, y, z respectively, then

$$U'=(3m+1)\,U, \text{ if } t=m\,(x+y+z).$$

[*In Exs.* 5 *to* 8 ω *denotes a primary cube root of* 1.]

5. Express the product

$$(x+y\omega+z\omega^2)\,(y+z\omega+x\omega^2)\,(z+x\omega+y\omega^2)$$

in the form $U+V\omega$, where U, V are scalar functions of x, y, z. Hence show that the expression

$$x^3+y^3+z^3+6xyz-3yz^2-3zx^2-3xy^2$$

is unchanged, if x, y, z are each increased by the same quantity t.

6. Prove that

$$(x+y)^3+(x\omega+y\omega^2)^3+(x\omega^2+y\omega)^3=3\,(x^3+y^3)$$

and

$$(x+y)^4+(x\omega+y\omega^2)^4+(x\omega^2+y\omega)^4=18x^2y^2.$$

7. If r_n denote the number of combinations of r things out of n, prove that $(x+y)^n+(\omega x+\omega^2 y)^n+(\omega^2 x+\omega y)^n$ is equal to

$$3\left\{x^n+3_n x^{n-3}y^3+6_n x^{n-6}y^6+\ldots+y^n\right\}$$

or $$3\left\{1_n x^{n-1}y + 4_n x^{n-4}y^4 + 7_n x^{n-7}y^7 + \ldots + 1_n xy^{n-1}\right\}$$

or $$3\left\{2_n x^{n-2}y^2 + 5_n x^{n-5}y^5 + 8_n x^{n-8}y^8 + \ldots + 2_n x^2y^{n-2}\right\},$$

according as n is of the form 3λ, $3\lambda - 1$, $3\lambda + 1$.

8. The sum of the squares of $x + y + z$, $x + \omega y + \omega^2 z$, $x + \omega^2 y + \omega z$ is $3(x^2 + 2yz)$, and that of the cubes is $3(x^3 + y^3 + z^3 + 6xyz)$.

9. Prove that

$$(x^3 + y^3 + z^3 - 3xyz)(x'^3 + y'^3 + z'^3 - 3x'y'z') = X^3 + Y^3 + Z^3 - 3XYZ,$$

if $X = xx' + yz' + y'z$, $Y = yy' + zx' + z'x$, $Z = zz' + xy' + x'y$.

10. If ω is a primary fifth root of 1, the sum of the squares of

$$x + y + z + u,\quad x\omega + y\omega^2 + z\omega^4 + u\omega^3,\quad x\omega^2 + y\omega^4 + z\omega^3 + u\omega,$$
$$x\omega^4 + y\omega^3 + z\omega + u\omega^2, \text{ and } x\omega^3 + y\omega + z\omega^2 + u\omega^4 \text{ is } 10(xz + yu).$$

11. If a chain of n links is coiled so that each link is inclined to the preceding at the angle $4/n^{\llcorner}$, the relative position of the two ends of the chain will be unaltered if the lengths of the links are all increased or decreased by the same amount.

CHAPTER VIII

INFINITE SERIES—CONVERGENCY AND DIVERGENCY

Before proceeding to those developments of our subject, in which the free use of infinite series is required, it is necessary to consider in some detail the general character of Algebraical Series, whether having a finite, or an infinite number of terms, in the light of Vector Algebra, and especially with reference to the question of their Convergency or Divergency. It is beyond the scope of the present work to give even a summary account [*] of the modern theory of convergency of series, as established by the investigations of Cauchy, De Morgan, Bertrand, and later writers, and this is the less necessary as, except in extreme cases (which may generally be dealt with by some special consideration applicable to the particular case), the ordinary simple tests of convergency or divergency are sufficient for the series which occur in Elementary Mathematics. These simple tests are derived from an examination of the character of a Geometric Series. We shall begin therefore by a complete discussion of such series.

1. *Geometric Series.*

A Geometrical Progression or Series is one in which each term is formed from the preceding by multiplication by the same factor. We may, without loss of generality, take the first term to be 1, then if z is the constant factor, the series is

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[*] For such an account see Chrystal's *Algebra*, Part II. Chap. xxvi. and the references in the historical note in the same chapter.

$$1 + z + z^2 + \dots + z^n + \dots$$

Let S_n denote the sum of the first n terms of this series, so that

$$S_n = 1 + z + z^2 + \dots + z^{n-1},$$

then $$S_n = 1 + z(1 + z + \dots + z^{n-2}) = 1 + z(S_n - z^{n-1}),$$

whence $$S_n = \frac{1 - z^n}{1 - z} = S - z^n S, \text{ if } S = \frac{1}{1 - z}.$$

The term, which we have here denoted by S, is independent of n and is a function, from which by the ordinary algebraical operation of division the series may be obtained to any number of terms. Being then essentially connected with the *form* of the series continued without limit as to the number of its terms, and without regard to the question of its *value* (whether finite or infinite), as depending on the value of z, S may properly be termed the "characteristic function" or "characteristic" of the series.

Let z be a vector whose tensor is r and versor i^u or $(\cos u + i \sin u)$,

then $$S_n = S - Sr^n i^{nu}.$$

Now if $r < 1$, by taking n sufficiently great r^n may be made as small as we please, and therefore ultimately when n is increased without limit, the term $Sr^n i^{nu}$ (since S is finite) becomes evanescent, and the limit of S_n is S. The infinite series is then said to be *convergent* and the finite quantity S, the limit of the sum of the terms as a greater and greater number are taken without limit, is called its *sum*: so that we may state that, when $r < 1$ and the series is therefore convergent, the sum of the series is equal to its characteristic.

If however $r > 1$, the term $Sr^n i^{nu}$ increases without limit with

n, and as the series has no finite limit, it is said to be *divergent* and the *characteristic* is no longer identical with the *sum.*

The case of $r = 1$ will require more detailed examination, but before proceeding with this, it will be well to reduce the forms of S and S_n to the standard forms of a vector or complex number.

2. *Reduction of S and S_n to vector forms.*

Since

$$S = \frac{1}{1 - ri^u} = \frac{1}{1 - r\cos u - ir\sin u},$$

therefore

$$S = \frac{(1 - r\cos u) + ir\sin u}{(1 - r\cos u)^2 + r^2\sin^2 u} = \frac{1 - r\cos u}{1 - 2r\cos u + r^2} + i\,\frac{r\sin u}{1 - 2r\cos u + r^2}$$

whence, if

$$\frac{\cos v}{1 - r\cos u} = \frac{\sin v}{r\sin u} = \frac{1}{(1 - 2r\cos u + r^2)^{\frac{1}{2}}},$$

$$S = \frac{\cos v + i\sin v}{(1 - 2r\cos u + r^2)^{\frac{1}{2}}} = \frac{1}{(1 - 2r\cos u + r^2)^{\frac{1}{2}}} \cdot i^v.$$

We have thus expressed the characteristic of the series both as the sum of a project and traject, and as the product of a tensor and vestor.

The series, being $1 + ri^u + r^2 i^{2u} + r^3 i^{3u} + \ldots\ldots$ may be put into the form

$$\begin{aligned} &1 + r\cos u + r^2\cos 2u + r^3\cos 3u + \ldots \\ &+ i(r\sin u + r^2\sin 2u + r^3\sin 3u + \ldots) \end{aligned}$$

which, if $r < 1$, is equal to S. Hence, equating projects and trajects, if $r < 1$,

$$1 + r\cos u + r^2\cos 2u + r^3\cos 3u + \ldots\,(ad\ inf.) = \frac{1 - r\cos u}{1 - 2r\cos u + r^2},$$

$$r\sin u + r^2\sin 2u + r^3\sin 3u + \ldots\,(ad\ inf.) = \frac{r\sin u}{1 - 2r\cos u + r^2}.$$

Farther

$$S_n = S - Sr^n i^{nu}$$

$$= \frac{\overline{1 - r\cos u} + ir\sin u}{1 - 2r\cos u + r^2} - r^n \frac{(1 - r\cos u + ir\sin u)(\cos nu + i\sin nu)}{1 - 2r\cos u + r^2}$$

or, after reducing the numerator of the second fraction,

$$S_n = \frac{\overline{1 - r\cos u} + ir\sin u}{1 - 2r\cos u + r^2}$$

$$- r^n \frac{(\cos nu - r\cos \overline{n-1}u) + i(\sin nu - r\sin \overline{n-1}u)}{1 - 2r\cos u + r^2}.$$

In another shape

$$S_n = S(1 - r^n \cos nu - ir^n \sin nu),$$

so that, if

$$\frac{\cos v_n}{1 - r^n \cos nu} = \frac{\sin v_n}{r^n \sin nu} = \frac{1}{(1 - 2r^n \cos nu + r^{2n})^{\frac{1}{2}}},$$

$$S_n = \frac{1}{(1 - 2r\cos u + u^2)^{\frac{1}{2}}} i^v . (1 - 2r^n \cos nu + r^{2n})^{\frac{1}{2}} i^{-v_n}$$

$$= \left(\frac{1 - 2r^n \cos nu + r^{2n}}{1 - 2r\cos u + u^2}\right)^{\frac{1}{2}} . i^{v - v_n}$$

Hence, equating projects and trajects as before,

$$1 + r\cos u + r^2 \cos 2u + \ldots + r^{n-1} \cos \overline{n-1}u$$

$$= \frac{1 - r\cos u}{1 - 2r\cos u + u^2} - r^n \frac{\cos nu - r\cos \overline{n-1}u}{1 - 2r\cos u + u^2}$$

$$\text{also} = \left(\frac{1 - 2r^n \cos nu + r^{2n}}{1 - 2r\cos u + r^2}\right)^{\frac{1}{2}} \cos(v - v_n)$$

$$r\sin u + r^2\sin 2u + \ldots + r^{n-1}\sin\overline{n-1}u$$

$$= \frac{r\sin u}{1-2r\cos u + r^2} - r^n\frac{\sin nu - r\sin\overline{n-1}u}{1-2r\cos u+u^2},$$

$$\text{also} = \left(\frac{1-2r^n\cos nu + r^{2u}}{1-2r\cos u + r^2}\right)^{\frac{1}{2}}\sin(v-v_n).$$

In the particular case of $r=1$, the conditions for the determination of v reduce to

$$\frac{\cos v}{2\sin^2\frac{u}{2}} = \frac{\sin v}{2\sin\frac{u}{2}\cos\frac{u}{2}} = \frac{1}{2\sin\frac{u}{2}} \text{ or } \cos v = \sin\frac{u}{2}, \sin v = \cos\frac{u}{2},$$

therefore $v = 1 - \frac{u}{2}$, and

$$S = \frac{1}{2\sin\frac{u}{2}}i^{1-\frac{u}{2}} = \tfrac{1}{2}\operatorname{cosec}\frac{u}{2}\left(\sin\frac{u}{2} + i\cos\frac{u}{2}\right) = \tfrac{1}{2} + i\,.\,\tfrac{1}{2}\cot\frac{u}{2}.$$

A similar reduction gives $v_n = 1 - \frac{nu}{2}$, $v - v_n = (n-1)\frac{u}{2}$,

$$\text{and } S_n = \left(\frac{1-\cos nu}{1-\cos u}\right)^{\frac{1}{2}} i^{\overline{n-1}\frac{u}{2}} = \frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\left\{\cos\overline{n-1}\frac{u}{2} + i\sin\overline{n-1}\frac{u}{2}\right\};$$

so that

$$1 + \cos u + \cos 2u + \ldots + \cos\overline{n-1}\,u = \frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\,.\,\cos\overline{n-1}\frac{u}{2}$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\frac{\sin\left(\overline{2n-1}\frac{u}{2}\right)}{\sin\frac{u}{2}}$$

$$\text{and} \quad \sin u + \sin 2u + \dots + \sin \overline{n-1}\, u = \frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}} \sin \overline{n-1}\, \frac{u}{2}$$

$$= \tfrac{1}{2} \cot \frac{u}{2} - \tfrac{1}{2} \frac{\cos \overline{2n-1}\, \frac{u}{2}}{\sin \frac{u}{2}}$$

The geometrical representation of the foregoing results will throw great light on their meaning.

3. *Geometrical Interpretation. Tensor ratio unity.*

It will be best to begin with the case, where $r = 1$, that is, with the series

$$1 + i^u + i^{2u} + i^{3u} + \dots$$

Let $\overline{OI_1}$ be the prime vector, and let $\overline{I_1I_2}$, $\overline{I_2I_3}$ . . , $\overline{I_{n-1}I_n}$ be the vectors representing the terms i^u, $i^{2u} \dots i^{\overline{n-1}u}$ respectively, and therefore all unit-vectors, their tensors being equal to OI, and each inclined to the preceding at the angle $u^{\llcorner}$: then

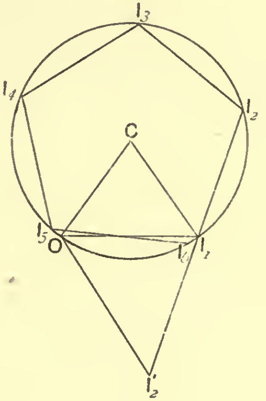
$$S_n = \overline{OI_1} + \overline{I_1I_2} + \dots + \overline{I_{n-1}I_n} = \overline{OI_n},$$

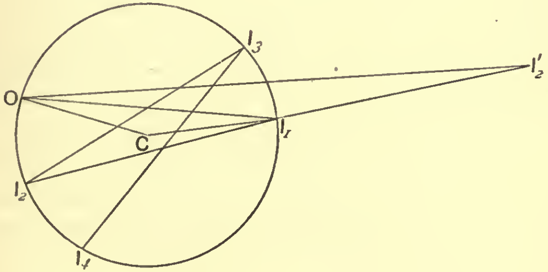
also $S = \dfrac{1}{1 - i^u}$, so that, if $\overline{I_2I_1}$ is produced to I_2' making $I_1I_2' = I_2I_1$, and therefore $\overline{OI_2}' = 1 - i^u$, $S = \dfrac{1}{\overline{OI_2}'}$.

Then on OI_1 describe the triangle OCI_1 similar to OI_1I_2', and we have $\overline{OC} = \dfrac{1}{\overline{OI_2}'} = S$.

Since $OI_1 = I_1I_2'$, $OC = CI_1$ and the angle $OCI = OI_1I_2' = u$, therefore $COI_1 = CI_1O = 1 - \dfrac{u}{2}$, and CI_1 bisects the angle OI_1I_2. Farther CI_1, I_1I_2 being respectively equal to CO_1, OI_1, and the included angles equal, $CI_2 = CI_1$ and it bisects the angle $I_1I_2I_3$.

Similarly CI_3, CI_4 . . . bisect the angles at I_3, I_4, . . . respectively, and therefore C is the centre of a circle passing through all the points O, I_1, I_2, I_3 . . .





Hence the characteristic of the series is the vector $\overline{OC}$ drawn from O to the centre of the circle on which lie all the extremities of the vectors $\overline{OI_1}$, $\overline{I_1I_2}$ . . . and of which those vectors are equal chords subtending the angle u at the centre, and forming a regular polygon inscribed in the circle, closed or unclosed, according as u is or is not commensurable with a right angle.

It is plain from the figure that $S = \overline{OC} = \frac{1}{2} \operatorname{cosec} \frac{u}{2}\, i^{1-\frac{u}{2}}$, as we found analytically in the preceding article.

Farther

$$S_n = \overline{OI_n} = \overline{OC} + \overline{CI_n} = \overline{OC} + \overline{CO}\, i^{nu},$$

since CI_n is obtained by turning CO about C through the angle nu: hence

$$S_n = \tfrac{1}{2} \operatorname{cosec} \frac{u}{2} \left(i^{1-\frac{u}{2}} - i^{1-\frac{u}{2}+nu} \right) = \tfrac{1}{2} \operatorname{cosec} \frac{u}{2} \,.\, i \,.\, i^{\overline{n-1}\frac{u}{2}} \left(i^{-\frac{nu}{2}} - i^{\frac{nu}{2}} \right)$$

$$= \operatorname{cosec} \frac{u}{2} \frac{i^{\frac{nu}{2}} - i^{-\frac{nu}{2}}}{2i} \,.\, i^{\overline{n-1}\frac{u}{2}} = \frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}} \,.\, i^{\overline{n-1}\frac{u}{2}},$$

as before proved.

It appears then that S_n is the sum of two unit vectors, one $(\overline{OC})$ fixed and the other adding u to its inclination for each increase of the number of the terms of the series by one, so that the sum of the series *fluctuates* about the central or mean value $\overline{OC}$, its tensor always lying between the limits 0 and $2OC$ or $\operatorname{cosec} \frac{u}{2}$. The series cannot therefore be said to be either convergent or divergent, since it neither approaches continually to a finite limit nor increases without limit. It may appropriately be termed a *fluctuating* series.

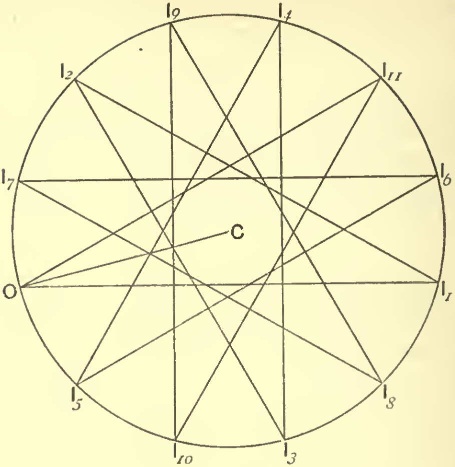
If u is a small angle, the radius OC is very large, and the limits of the fluctuation are widely apart, while, if $u = 0$, the circle becomes in the limit the straight line OI infinitely extended through I, and the series being then $1 + 1 + 1 + 1 + \ldots$ is obviously divergent. As u increases the radius of fluctuation diminishes, until when $u = 2^{\llcorner}$, C coincides with the middle point of OI, and $S_n = \frac{1}{2} + \frac{1}{2} i^{nu} = \frac{1}{2} + (-1)^n = 0$ or 1, as n is odd or even, as is obvious from the series itself, which becomes in this case the *oscillating* series $1 - 1 + 1 - 1 + \ldots$

As u increases beyond $2^{\llcorner}$, C passes to the other side of OI, and the radius of fluctuation increases till at $u = 4^{\llcorner}$, it again becomes infinite, and then, passing to infinity on the other side of OI, repeats the same cycle of variation, if u is supposed to go on increasing.

Hence a geometric series whose common ratio is a given unit vector, that is, a series of unit vectors, each inclined to the preceding at the same angle, is neither convergent nor divergent, but fluctuating. The characteristic of the series is the radius of the circle of fluctuation, and is the mean value of the sum of its terms, differing from the sum of any number of terms by one of the vector radii of the circle. The tensor (or modulus) of each sum is the length of some chord of this circle and therefore ranges between 0 and its diameter. In the limiting case, when the angle between the vectors vanishes, the series becomes scalar and all its terms positive, the radius of the circle of fluctuation infinite and the series divergent. The series also becomes scalar if the angle is two right angles, but its terms are then alternately positive and negative, the diameter of the circle of fluctuation is the least possible and equal to unity, and the series is *oscillating*. Between these two extremes the circle of fluctuation may be of any magnitude, being larger or smaller as the angle between the vectors is nearer to zero or two right angles.

The student should construct diagrams for a few cases, where u is a simple fraction, *e.g.* $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{3}$, &c. The diagram below corresponds to the case of $u = \frac{5}{3}^{\llcorner}$ or 150°, and it will be observed that the sum of the series has twelve distinct values as the number of terms is increased, the same values being repeated over and over again continually in the same order.

It is easy to show that if $u = \dfrac{p}{q}$, where p is prime to q, the sum has either $4q$ or $2q$ or q distinct values, repeated over and over in periods.



4. *Tensor ratio* < 1. *Series convergent.*

Next consider the series

$$1 + ri^u + r^2 i^{2u} + \ldots, \text{ when } r < 1.$$

Take

$$\overline{OP_1} = \overline{OI_1} = 1,\ \overline{P_1P_2} = r.\overline{I_1I_2} = ri^u,\ \overline{P_2P_3} = r^2.\overline{I_2I_3} = r^2 i^{2u},$$
$$\overline{P_3P_4} = r^3.\overline{I_3I_4} = r^3 i^{3u},$$

and so on; then the polygon $OP_1P_2P_3 \ldots$ will have its successive sides parallel respectively to those of $OI_1I_2I_3 \ldots$, but of lengths diminishing continually in geometrical progression, and therefore lying within the latter, and so inside the circle of fluctuation, as represented in the diagram.

Then

$$S_n = \overline{OP_1} + \overline{P_1P_2} + \overline{P_2P_3} + \ \ldots\ + \overline{P_{n-1}P_n} = \overline{OP_n},$$

and the characteristic of the series

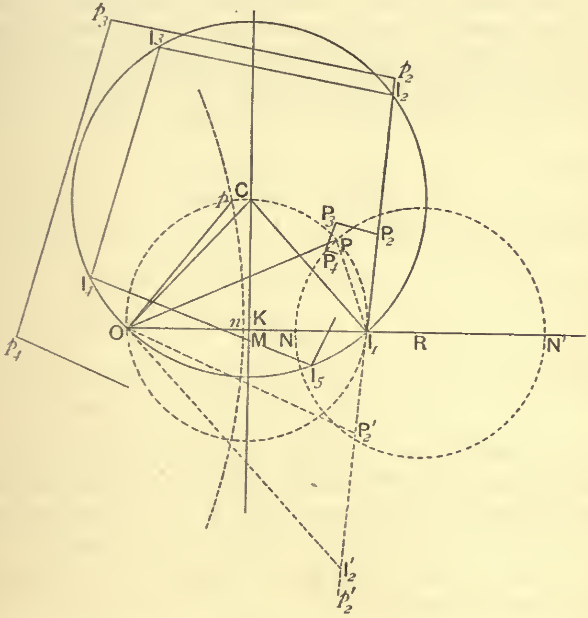
$$S = \frac{1}{1 - ri^u} = \frac{1}{\overline{OI_1} - \overline{I_1P_2}}.$$

On $I_1I'_2$ take $I_1P'_2=P_2I_1$, then

$$S=\frac{1}{\overline{OI_1}+\overline{I_1P'_2}}=\frac{1}{\overline{OP'_2}}.$$

On OI_1 describe the triangle OPI_1 similar to $OI_1P'_2$, then

$$\overline{OP}=\frac{1}{\overline{OP'_2}}=S.$$



Since the angle $OPI_1=OI_1P'_2=u$, P lies on the circumference of the circle circumscribing OCI. Also $OP:PI_1::OI_1:I_1P'_2::1:r$, so that the distances of P from O and from I are in the ratio of 1 to r: hence if N, N' are points which divide the line OI_1 internally and externally in the ratio of 1 to r, it is well known from Elementary Geometry that P is on the circumference of the circle of which NN' is a diameter. Hence P may be determined by the intersection of two circles; one of them, which we may term the *u-circle*, passing through O and I, so that OI_1 cuts off a segment containing an angle equal to u and so dependent on u *only*, the other, which we may term the *r-circle*, described on the line joining the points which divide OI_1 internally and externally in the ratio of 1 to r, as diameter and so dependent on r *only*.

It is noteworthy that the two systems of circles, the r-circles or *equitensal* circles and the u-circles or *equiclinal* circles, are both coaxal systems determined by the two points O and I_1, the former being a non-intersecting system having the perpendicular to OI_1 through its middle point as its radical axis and O, I_1 its limiting points, and the latter a system passing through the points O and I_1 and so having OI_1 as its radical axis. They are therefore systems which cut one another *orthogonally*. Cf. Casey, *Sequel to Euclid*, Book vi., Section v.

Farther

$$\overline{S}_n = \overline{OP_n} = \overline{OP} + \overline{PP_n},$$

and we have seen that

$$S_n = S - Sr^n i^{nu},$$

and

$$S = \overline{OP},$$

therefore

$$\overline{PP_n} = \overline{PO}\, r^n i^{nu},$$

which, r being less than 1, decreases without limit as n increases, so that ultimately, when n is infinite, P_n coincides with P. Hence the series is truly convergent if $r < 1$, the angles of its representative polygon getting nearer and nearer without limit to P, and its sum *ad infinitum* is the vector $\overline{OP}$, which is also the characteristic of the series.

It will be instructive to examine the different positions of P, both when r is given and u changes, and when u is given and r changes.

Since N, N' divide OI_1 in the ratio of 1 to r,

$$ON = \frac{1}{1+r}, \quad NI_1 = \frac{r}{1+r}, \quad ON' = \frac{1}{1-r}, \quad I_1N' = \frac{r}{1-r},$$

$$RN = RN' = \frac{r}{1-r^2}, \quad OR = \frac{1}{1-r^2},$$

R being the middle point of NN' and therefore the centre of the *r-circle*.

First, when r is given.

If u is very small, the u-circle is very large and intersects the r-circle very near to N', which is the limiting position of P, when u vanishes and the series therefore has all its terms scalar and positive. The sum $\overline{OP}$ has then in this case its tensor the greatest possible and equal to $\frac{1}{1-r}$, the value found in scalar algebra for the sum of a geometric series.

As u increases P moves on the *r-circle* from N' towards N and coincides with N when $u = 2^{\llcorner}$. The series then becomes the alternating series $1 - r + r^2 - r^3 + \ldots$, whose sum is $\frac{1}{1+r}$, and the tensor OP then has its least value.

When u increases beyond 2 right angles, P moves on the other half of the circumference from N towards N'.

If OP touches the r-circle, $\sin POI_1 = PR/OR = r$: hence the inclination of OP to OI_1 lies between the limits $+\sin^{-1} r$ and $-\sin^{-1} r$, and therefore between narrower or wider limits as r is less or greater.

Next, when u is given.

If r is very small, the r-circle is small and N, N', P are all very near to I_1, and the sum of the series is very near to 1, its first term, to which it reduces, if $r = 0$.

As r increases, N approaches to M the middle point of OI_1 and N' recedes from I, as also does R, so that the r-circle increases and P moves along the u-circle away from I, until in the limit, when $r = 1$, N coincides with M, N' is at infinity and the r-circle becomes the perpendicular to OI_1 through M, and P coincides with C and the series ceases to be convergent and becomes fluctuating.

Hence it appears that, for a suitable value of u and any value of $r < 1$, P may occupy any position on the I side of the perpendicular to OI through its middle point, and the series is convergent.

5. *Tensor ratio* > 1. *Series Divergent.*

Making a construction similar to that in the preceding article, the polygon $Op_1\,p_2\ldots$ will have its successive sides parallel to those of $OI_1I_2\ldots$, but of lengths continually increasing and therefore lying outside the latter, and so outside of the circle of fluctuation, as represented in the diagram.

The position of P will be determined as before by the intersection of the u-circle and the r-circle, but now, r being > 1, N and N' will lie on the O side of M, and the r-circle will intersect the u-circle on the O side of the perpendicular to OI through M. Also since $\overline{PP_n} = \overline{PO} \,.\, r^n\, i^{nu}$, PP_n increases without limit as n increases, and P_n continually recedes farther and farther from P. Hence the sum of the series has no limit as the number of terms is increased, and the series is therefore divergent. In this case, although the characteristic of the series $(= OP)$ is still definite, it cannot be said to be the *sum* of the infinite series, as it was in the case of convergency, when $r < 1$.

6. *Summary of Results.*

The foregoing results may now be conveniently summed up as follows :—

Regarding the terms of a geometric series as a series of an infinite number of links of a chain, if the chain is stretched out to its full extent so that the links are all in the same straight line, it extends to infinity, or the series is divergent, if the ratio (r) of the length of each link to the preceding is greater than or equal to unity, but extends only to a finite distance, or the series is convergent, if r is less than 1.

If now the chain is coiled round so that each link is inclined to the preceding at the same fixed angle (u), it still extends in wider and wider circuits to infinity, if $r > 1$. But if $r = 1$, or the links are all equal, the coils all lie within a certain circle, the circle of fluctuation, the extremities of all the links being on the circumference of the circle. This circle is very large, if the inclination of each link to the preceding is small, but diminishes as this angle increases, until finally when it becomes 2 right angles or a straight angle, the links are all superposed on the first, which is then the diameter of the circle of fluctuation at its minimum. The series is then an *oscillating* scalar series, its terms being alternately positive and negative, whose sum oscillates between the two values 0 and 2. Hence if $r = 1$, the series is fluctuating, and its sum has no single limit but fluctuates between a finite or infinite number of finite values within the limits 0 and the diameter of the circle of fluctuation, except in the single extreme case, when the angle is zero or the chain is fully extended, in which case the sum is infinite or the series divergent.

If however r is less than 1 or the successive links continually diminish from the first, the fully extended chain extends only to a finite distance, and when coiled as before, its extremity continually approaches without limit to a definite point on that half of the plane which lies on the I side of the line which bisects the prime vector (or first link) at right angles. The sum of the series then has always a definite limit, or the series is always convergent. When the chain is most extended, its length is $1/1-r$, and when most compactly packed by the successive links being superposed in opposite senses on the first, its extremity is distant from the origin by $1/1+r$; in other positions this distance is intermediate between these extremes.

Whatever be the value of r, the characteristic of the series has a definite value (OP) corresponding to a definite position of P on the plane, which is on the O side or the I side of the perpendicular to OI through its middle point according as r is greater or less than 1. In the latter case only however is the sum finite, and therefore quantitatively equal to the characteristic.

Infinite Series in general.

7. *Definitions.*

The general form of an infinite series of complex terms or vectors may be written thus:

$$r_1 i^{u_1} + r_2 i^{u_2} + \ldots + r_n i^{u_n} + \ldots$$

where $r_1, r_2 \ldots$ being moduli or tensors are positive numbers, and r_n, u_n are functions of the integral number n, and, it may be, of one or more other variables.

If the sum of the first n terms of such a series has a *definite finite limit*, when n is increased without limit, the series is said to be *convergent*.

If the modulus (or tensor) of the sum of n terms increases without limit with n, the series is said to be *divergent*.

It may be however, as we have seen in the foregoing discussion of a geometric series, that the sum of n terms is always finite but yet has no *definite limit*, when n is increased without limit, either because the modulus (or tensor) of the sum becomes indefinite, when n is increased without limit (e.g. $\sin nu$), or because its versor is indefinite through its inclination becoming infinite. In such a case we term the series *fluctuating*, and when its terms reduce to scalars, *oscillating*.

8. *General properties of Infinite Series.*

A few of the simpler general properties of infinite series may now be stated. Many of these become almost self-evident, if we regard each vector-term as a *link* of definite length (the tensor of the term) and the series as a *chain* of an infinite number of such links. If the links have all the same direction; that is, if $u_1, u_2 \ldots u_n \ldots$ are all equal, the chain is extended to its full length, and if in this state the length of the chain is finite, the distance between its extremities must in all states remain finite, however the chain be folded or coiled, or whatever be the values of $u_1, u_2 \ldots$

It is thus obvious that if the series $r_1+r_2+\ldots+r_n+\ldots$ is convergent, still more is the general series $r_1 i^{u_1}+r_2 i^{u_2}+\ldots+r i^{u_n}+\ldots$ convergent. The series is in this case said to be *absolutely convergent.*

A chain of an infinite number of links can obviously be finite when fully extended, only if from and after a certain point on the chain (if not from the beginning) the successive links continually diminish so that the n^{th} link may be made as small as we please by taking n sufficiently great. Thus the series cannot be *absolutely convergent* unless the limit of r_n, when n is infinite, is zero.

The converse however is not true, for the series *may be* divergent although $\mathrm{lt}_{n=\infty}(r_n)=0$ or the length of the chain may be infinite, although the links become ultimately infinitely small. A well-known instance of this is the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots \text{ or } \Sigma_1^\infty\left(\frac{1}{n}\right),$$

which, by grouping the terms thus,

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+\ldots$$

is obviously greater than

$$1+\tfrac{1}{2}+\tfrac{2}{4}+\tfrac{4}{8}+\tfrac{8}{16}+\ldots \text{ or } 1+\tfrac{1}{2}+\tfrac{1}{2}+\ldots,$$

of which the sum is evidently infinite.

When such a chain is folded or coiled, it will obviously depend on the law of folding, whether it extends to a finite or to an infinite distance. In other words, when $\mathrm{lt.}_{n=\infty}(r_n)=0$, but $\Sigma_1^\infty(r_m)$ is infinite, it will depend on the law determining the values of $u_1, u_2, u_3 \ldots$ whether the series $\Sigma_1^\infty(r_n i^{u_n})$ is convergent or divergent.

A chain may be most compactly coiled by folding each link over the preceding one so that the links of odd order have one direction and sense and those of even order exactly the contrary. It is obvious that in this case, if the links are ultimately infinitely small, the distance between its extremities is finite. In other words, if $u_1 = u_2 = u_3 = \ldots$ and $u_2 = u_4 = u_6 = \ldots$, but $u_2 = 2 + u_1$, so that the series reduces to

$i^{u_1}(r_1 - r_2 + r_3 - r_4 + \ldots)$ or, if $u = 0$, to $r_1 - r_2 + r_3 - r_4 + \ldots$, the series has a definite finite limit.

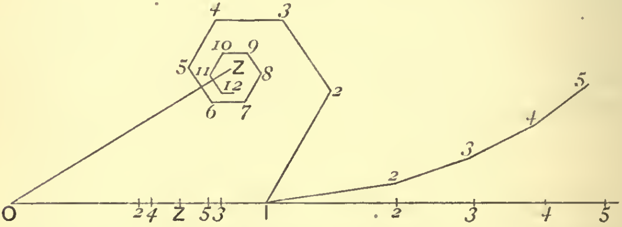
Hence *an infinite alternating scalar series* (that is, *one whose terms are alternately positive and negative scalars*) *is convergent, if the successive terms diminish without limit.*

Between the extreme cases of

$$\Sigma_1^\infty (r_n) \text{ divergent and } \Sigma_1^\infty \{(-1)^n r_n\} \text{ convergent,}$$

in the absence of any general criterion, the convergency or divergency of $\Sigma_1^\infty (r_n i^{u_n})$ must be investigated for each particular case.

[As an instance, take the series $1 + \frac{1}{2}i^u + \frac{1}{3}i^{2u} + \frac{1}{4}i \quad + \ldots$ in which $r_n = \dfrac{1}{n}$, so that $r_n = 0$ ultimately, when n is infinite.



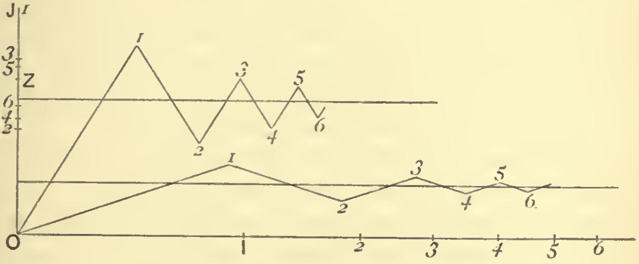
When $u = 0$, the series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ which, as we have seen, is divergent. This is represented in the diagram by the links *OI*, *I*2, 23, 34 . . . all lying in the same straight line and the chain extending to infinity.

When u is a small angle, (in the diagram nearly 10°), the chain is seen extending in a regularly bent line, each link inclined to the preceding at the same angle and for the few earlier links getting further away from 0 to a great distance, but then bending round and obviously ultimately tending to a point at a finite though large distance from 0.

When $u = \frac{2}{3}\llcorner$ or 60°, the chain is coiled as in the figure, and its ultimate extremity is Z, such that (as we shall afterwards be able to prove) $\overline{OZ} = \frac{\pi}{3} \cdot i^{\frac{1}{3}}$: that is, $OZ = \frac{1}{3}\pi \cdot OI$ and the angle $IOZ = \frac{1}{3}\llcorner$ or 30°.

When $u = 2$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$, and as shown in the diagram, the chain is coiled in its most compact form, all the links lying on OI, the odd links in the sense OI, and the even links in the opposite sense, and the end therefore ultimately approaching the limit Z, such that (as it will be easy to show later on) $OZ = \log_e 2 = \cdot 693 \ldots$

Hence the series above is convergent for all values of u, except only for the extreme case, where $u = 0$ or $4\lambda\llcorner$, λ being a positive integer.



As a second instance, take the series $i^u + \frac{1}{2}i^{-u} + \frac{1}{3}i^u + \frac{1}{4}i^{-u} + \ldots$

This like the former reduces to $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ when $u = 0$; and when $u = 1$, to $i(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots)$. In the former case the series is divergent, in the latter convergent. They are represented in the diagram by the chain stretched out to its full length along OI and coiled most compactly along OJ. For intermediate values of u, it is obvious that the projects of the links form the series $\cos u(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots)$, which is divergent, and the trajects the series $\sin u\,(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots)$ which is convergent, and therefore the series as a whole is divergent. Hence the series is in all cases divergent, except only when $u = 1^{\llcorner}$ or $2\lambda + 1^{\llcorner}$, for which values it is convergent, the divergence however being towards a point at an infinite distance along a line parallel to OI at the distance $\log_\epsilon 2 \,.\, \sin u$ from it.

In these two examples if we regard the infinite series as functions of u, the first is at once seen to be a *periodic function* which, commencing with an infinite value when $u = 0$, is finite and has a continually diminishing modulus or tensor from $u = 0$ to $u = 2$; a definite finite value, when $u = 2$, (to be afterwards shown to be $\log_\epsilon 2$); and then an increasing modulus from $u = 2$ to $u = 4$, when it again becomes infinite: the same series of changes is then repeated over and over again in successive periods. We may expect therefore that, if an equivalent finite expression for the infinite series can be found, it will be a periodic function of $\frac{u}{2}$, which is continuous, except $\left(\text{like } \cot\frac{u}{2}\right)$ for the values $u = 0$ or 4λ. This will later be shown to be the case. The other series is a function of u, which is finite only when $u = 2\lambda + 1$, and is therefore essentially discontinuous, although its separate terms are continuous, that is, change gradually by infinitely small increments for infinitely small increments of u.]

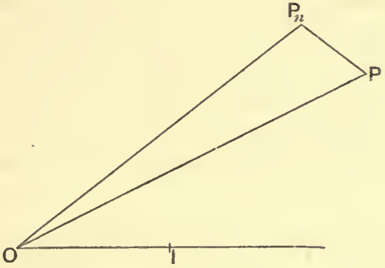
If r_n does not vanish but has a finite limit, when n is infinite, although the series of moduli (or tensors) $r_1 + r_2 + \ldots r_n + \ldots$ is obviously divergent, the vector series $r_1 i^{u_1} + r_2 i^{u_2} + \ldots$ may, as we have seen in the case of the geometric series, when the common ratio is a unit vector or i^u, have a sum which is finite, but indeterminate, or, in other words, the series though not convergent may be *fluctuating*.

9. *Rate of Convergency of a Convergent Series.*

Let S_n, represented by the vector $\overline{OP_n}$, denote the sum of n terms of a series, and S, represented by $\overline{OP}$, the finite limit of

S_n, when n is increased without limit, then R_n, the remainder of the series after n terms, is represented by $\overline{P_nP}$ and is equal to $S-S_n$.

Let $R_n=\epsilon.S.i^\nu$ where ϵ is a positive number, the modulus of the ratio R_n/S or the tensor of $\overline{P_nP}/\overline{OP}$: then after a certain value of n (if not from the beginning of the series) ϵ must continually diminish as n increases and ultimately vanish, when n is infinite.



According as ϵ is greater or less for a given value of n, or according as n is greater or less for a given value of ϵ, the convergency of the series may be said to be less or more rapid, or more or less slow. Hence the magnitude of ϵ when n is given, or that of n when ϵ is given, may be taken as a measure of the *slowness* of convergency of the series. As an illustration, consider the geometric series discussed in § 1—4.

In this case $R_n=S-S_n=Sr^ni^{nu}$, whence $\epsilon=r^n$ and $\nu=nu$. Then if r is a small fraction, for a given value of n, ϵ is very small, and the convergency is rapid: as r increases, for the same value of n ϵ becomes larger and the convergency slower: until when r is nearly $=1$, ϵ becomes large, or if ϵ be a given small fraction, n is very large and the convergency excessively slow; and ultimately it may be said to be *infinitely slow*, when $r=1$ and the series ceases to be convergent by becoming fluctuating. Since ϵ depends on r only and not on u, all the geometric series which have the same value of r have the same rate of convergency, whatever be the value of u. In the diagram of § 4.

while P moves along the equitensal circle NPN' the rate of convergency remains constant, while if P be the centre of a small circle whose radius varies as OP, in order that P_n, $P_{n+1}, \ldots$ may be within that circle, n must be greater and greater, (or more and more links of the chain must be taken) as P moves on the equiclinal circle I_1PC from I_1 to C, becoming infinite as P comes up to C; or along the circle between I_1 and C the slowness of convergency varies from 0 to ∞.*

It should be observed that, since $r = nu$, r becomes infinite with n (unless $u = 0$) and therefore the versor i^r indeterminate, corresponding to what is in itself obvious that $\overline{P_nP}$ has no ultimate determinate direction. Also that when $r = 1$ and the convergency ceases, $\overline{P_nP}$ is ultimately finite instead of evanescent, but has no ultimate determinate direction, or, as we have seen, the series is fluctuating.

10. *Continuity or Discontinuity of Infinite Series.*

We have seen, in the instances discussed in § 8, that the sum of an infinite series of terms, which are continuous functions of a variable, may be discontinuous for certain values or beyond certain limiting values of the variable. It is important therefore to examine more generally under what conditions such discontinuity may occur, or rather under what conditions continuity may be certainly predicated.

Let $z_1, z_2, z_3 \ldots z_n$ be a series of complex numbers or vectors, and let $Z_n = z_1 + z_2 + \ldots + z_n$, and let Z be the limit of Z_n, when $n = \infty$, the series being convergent. Also let $\zeta_1, \zeta_2, \zeta_3 \ldots \zeta_n$ be a like series of continuous functions of the scalar† x, which reduce to $z_1, z_2, \ldots z_n$ for the particular value $x=a$, then, if $Z'_n=\zeta_1+\zeta_2+\ldots+\zeta_n$ and Z' is its limit, when n is increased without limit, the series being in general convergent, the question is: whether we may assert that $Z = Z$, when $x=a$?

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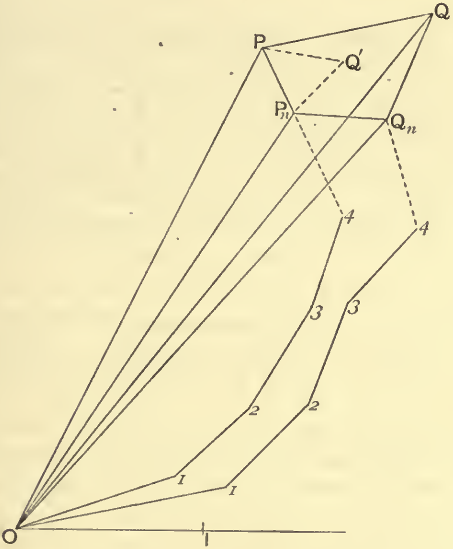
* Continuing this discussion beyond the value $r = 1$, it would appear that the value of ϵ being greater, for a given value of n, the greater r is, this value would properly measure the rate of *divergency* of the sum of the series from the value of its characteristic function.

† For the sake of simplicity we limit the discussion in the text to functions of a *scalar*, though properly understood the result is equally true of functions of a *complex number* or *vector*. The discussion however of the nature of functions generally of a complex number and their continuity is beyond the scope of this work. On this point the student may consult Hobson's *Trigonometry*, pp. 247—251.



Stated geometrically, let $\overline{OP}=Z$, $\overline{OP_n}=Z_n$, $\overline{OQ}=Z'$, $\overline{OQ_n}=Z'_n$, then, when $x=a$, by hypothesis Q_n coincides with P_n; can we then say that Q coincides with P? In other words, does $\overline{PQ}$ diminish without limit, as x approaches without limit to a and therefore Q_n to P_n? Observe that we proceed from Z'_n to Z by *first* taking $x=a$ which reduces Z'_n to Z_n (brings Q_n to P_n) and *then* making n infinite (bringing P_n to P). On the other hand we proceed from Z'_n to Z' (from Q_n to Q) by *first* making n infinite: shall we then reduce Z' to Z (bring Q to P), by taking $x=a$?

An instance will show that this is not necessarily the case. Take the function $\dfrac{ax^2+bxy+cy^2+mx+ny}{a'x^2+b'xy+cy^2+m'x+n'y}$ regarding x, y as quite independent. Putting $y=0$, this reduces to $\frac{ax^2+mx}{a'x^2+m'x}$ or $\frac{ax+m}{a'x+m'}$, and then putting $x=0$, it farther reduces to $\frac{m}{m'}$. On the other hand first putting $x=0$, and then $y=0$, we obtain successively $\frac{cy+n}{cy+n'}$ and $\frac{n}{n'}$. Again, if we suppose y and x connected by the equation $y=kx$, the fraction reduces to $\frac{(a+bk+ck^2)x+(m+nk)}{(a'+b'k+ck^2)x+(m'+n'k)}$, so that on this supposition, when $x=0$ and therefore $y=0$, its value is $\frac{m+nk}{m'+n'k}$.

Thus it appears that, as long as x, y are quite independent, the value of the function for $x=0$ and $y=0$ is quite indeterminate, and can only be rendered determinate by assuming a particular relation between x and y.

For a given value of n, let $Z-Z_n$ or $\overline{P_nP}=\epsilon\,.\,i^\nu Z$ or $\epsilon i^\nu\overline{OP}$, then, since the series $z_1+z_2+\ldots$ is convergent, $\overline{OP}$ is finite and ϵ may be made as small as we please by taking n sufficiently great. For the same value of n, let $Z'-Z'_n$ or $\overline{Q_nQ}=\epsilon' i^{\nu'}Z'$ or $\epsilon' i^{\nu'}\overline{OQ}$, and suppose the series $\zeta_1+\zeta_2+\ldots$ to remain convergent as x approaches the limit a, and also at the limit, when $x=a$, then ϵ' also may be made as small as we please by taking n sufficiently great. Now when $x=a$, Z'_n becomes equal to Z_n or Q_n coincides with P_n: suppose that Q then comes to Q' so that $\overline{OQ'}=\text{lt.}_{x=a}Z'$, then PQ' cannot be greater than P_nP+P_nQ' or $\epsilon\,.\,OP+\epsilon'\,.\,OQ'$: but both ϵ, ϵ' may be made as small as we please by taking n sufficiently great, and therefore PQ' cannot have any finite value, so that Q' must coincide with P, and $\text{lt.}_{x=a}\,Z'=Z$.

If however as x approaches to the value a, the rate of convergency of the series Z' becomes slower, and at last, when $x=a$, infinitely slow, we cannot assert that ϵ' can be made as small as we please for a sufficiently great value of n and therefore PQ' does not necessarily vanish, and $\text{lt.}_{x=a}\,Z'$ may be different from Z.

Hence

If an infinite series, whose terms are continuous vector functions of a scalar variable x, is convergent for values of x continually *approaching to the particular value $x=a$, for which its terms reduce to those of a definite convergent series, the limit of its sum, when $x=a$, cannot differ from the sum of the latter series, unless the first becomes infinitely slowly convergent when $x=a$.*

The foregoing discussion is essentially that given by Prof. Sir G. Stokes (*Math. and Phys. Papers*, i. p. 280) in a paper "On the critical values of the sums of Periodic Series," read before the Cambridge Philosophical Society, Dec. 6, 1847. This paper (I believe) contained the first published statement of the dependence of the discontinuity of an infinite series on the infinitely slow convergence of the series for a particular value of the variable. The same result was published in a note by *Seidel* in 1848 in the *Transactions of the Bavarian Academy.*

The following example adapted from one given by Stokes will illustrate the argument of this article.

Consider the two series

$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\,.\;.\;.+\frac{1}{n\,(n+1)}+\,.\;.\;.$$

$$\frac{x}{1+x}+\frac{x}{(1+x)\,(1+2x)}+\frac{x}{(1+2x)\,(1+3x)}+\,.\;.\;.$$
$$+\frac{x}{(1+\overline{n-1}x)\,(1+nx)}+\,.\;.$$

They may be put into the forms

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\,.\;.\;.+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\quad.\;.\;.\;.\;.\;(1)$$

$$\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{1+2x}\right)+\,.\;.\;.$$
$$+\left(\frac{1}{1+\overline{n-1}x}-\frac{1}{1+nx}\right)+\quad.\;.\;.\;.\;(2)$$

whence it is seen that their sums to n terms are respectively $1-\dfrac{1}{n+1}$ and $1-\dfrac{1}{1+nx}$, and their sums *ad infinitum* are each 1 as long as x differs by however small a quantity from 0.

If we add the two series, term by term, we have the infinite series,

$$\frac{1+3x}{2\,(1+x)}+\frac{1+9x+2x^2}{2\,.\,3\,(1+x)\,(1+2x)}+\,.\,.\,.$$

$$+\frac{1+(n^2+3n-1)\,x+n\,.\,\overline{n-1}\,x^2}{n\,.\,(n+1)\,(1+\overline{n-1}\,x)\,(1+nx)}+\,.\,.\,.\,.\quad(3)$$

which is convergent and of which the sum must therefore be 2, whatever the value of x. If however in this last series we put $x=0$, the series is reduced to the series

$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\,.\,.\,.\,+\frac{1}{n\,(n+1)}+\,.\,.\,.\,,$$

whose sum is 1. Hence there is discontinuity when $x=0$. In fact, using P_n, Q_n as in the proof above, while

$$P_nP=\frac{1}{n+1}\qquad Q_nQ=\frac{1}{n+1}+\frac{1}{1+nx},$$

so that P_nP, Q_nQ differ by the finite quantity $\frac{1}{1+nx}$, which is nearly 1, when x is small. Hence for a given small value for x (in order to get within a given distance ϵ from the limit), the number of terms required to be taken in the series (3) would be much greater than in the series (1), or the former converges much more slowly than the latter. As x diminishes, the difference in these numbers of terms is greater without limit, or the series (3) converges *infinitely slowly* close to the critical value 0 of x.

11. *The Fundamental Laws of Algebra in relation to Infinite Series.*

The Associative and Commutative Laws of Algebra are not necessarily true for the terms of an infinite series, as they are for those of a finite expression. The *order of succession* of the terms is, in fact, of the essence of an infinite series, so that if that order is changed, the series thus obtained has not necessarily the same value, or does not represent the same function, as the original series.

To show this, an instance will suffice. Consider the series $z_1+z_2+z_3+\ldots+z_n+\ldots$, and form a second series by displacing the terms of even order so that *two* of odd orders are followed by *one* of an even order, and let the series be $v_1 + v_2 + v_3 \ldots$, where $v_1 = z_1 + z_3 + z_2$, $v_2 = z_5 + z_7 + z_4 \ldots$, and generally $v_n = z_{4n-3} + z_{4n-1} + z_{2n}$. Then

$$\Sigma_1^n(v_m) = \Sigma_1^{4n}(z_m) - (z_{2n+2} + z_{2n+4} + \ldots + z_{4n}),$$

and therefore we cannot assert that $\Sigma_1^\infty(v_m) = \Sigma_1^\infty(z_m)$, unless $\text{lt.}_{n=\infty}\,(z_{2n+2} + z_{2n+4} + \ldots + z_{4n}) = 0$.

Now if the z series is *absolutely* convergent, (that is, if $\Sigma_1^\infty r_n$ is finite,) $\Sigma_1^\infty z_n$ or $\Sigma_1^\infty(r_n i^{u_n})$ is in all cases finite, the remainder after n terms vanishing ultimately when n becomes infinite, so that in this case the limit above is always 0, and the v series has the same sum as the z series, and the commutative and associative laws do hold.

If however the series is not absolutely convergent, so that $\Sigma_1^\infty r_n$ is divergent, and the convergency of $\Sigma_1^\infty z_n$ or $\Sigma_1^\infty(r_n i^{u_n})$ depends on the law of the values of $u_1, u_2, u_3 \ldots$, it may be that the limit of $z_{2n+2} + z_{2n+4} \ldots + z_{4n}$ does not vanish when n is infinite, and then the original series and the altered series have different sums.

Recurring to the illustration of a chain, suppose the chain to consist of alternately white and black links: then if the order of the links is altered so as to have two white links followed by one black link, it will be obvious that at the end of n such triple links there will be n black links displaced which have not been used, namely, those following the $(2n)^{\text{th}}$ link as far as the $(4n)^{\text{th}}$. Now if the chain (though the number of its links is infinite) is of finite length when fully stretched (the case of an *absolutely* convergent series), both the links themselves and the sum of any number of them after the n^{th} continually diminish as n is increased and ultimately vanish when n is infinite, so that the unused black links may ultimately be neglected. But if the chain, when fully extended, is infinite in length, although the successive links diminish and ultimately vanish, it may be that the n unused black links form a chain of finite length, or even extending to infinity unless suitably folded or coiled.

As an instance take the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ or $\Sigma_1^\infty\left(\frac{i^{2n}}{n}\right)$, which (§ 8, p. 186) is convergent and has in fact the sum $\log_e 2$, while $\Sigma_1^\infty\left(\frac{1}{n}\right)$ is (as we have seen) infinite. Then

$$v_n = \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = \frac{8n-3}{(4n-3)(4n-1)2n},$$

and the new series is

$$\frac{5}{1.3 \times 2} + \frac{13}{5.7 \times 4} + \frac{21}{9.11 \times 6} + \ldots,$$

a convergent series, of which the first n terms exceed $3n$ terms of the original series by $\frac{1}{2n+2} + \frac{1}{2n+4} + \ldots + \frac{1}{4n}$. Now this quantity is obviously less than $\frac{n}{2n+2}$ and greater than $\frac{n}{4n}$, and therefore ultimately, when $n = \infty$, it has a finite value between $\frac{1}{4}$ and $\frac{1}{2}$, so that the sum of the second series differs by a finite quantity from that of the first, although it was derived from it merely by an alteration of the order and grouping of the terms.

Again consider the series $\frac{0}{1} - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \ldots$, which is shewn below [§ 15, (6)] to be an *oscillating* series, and, without altering the order, group the terms in pairs: we then have the series

$$-\frac{1}{1.2} - \frac{1}{3.4} - \frac{1}{5.6} \ldots - \frac{1}{(2n+1)(2n+2)} \ldots,$$

which is convergent. In fact, since

$$\frac{2n}{2n+1} - \frac{2n+1}{2n+2} = \left(1 - \frac{1}{2n+1}\right) - \left(1 - \frac{1}{2n+2}\right)$$

$$= 1 - 1 - \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right),$$

the first series is the sum of the second (convergent) series and the oscillating series

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

Thus the law of Association of Terms must not be used absolutely in the case of all infinite series.

The foregoing instances are sufficient to shew that, unless an infinite series is *absolutely* convergent, the laws of Association and Commutation of Terms can only be applied upon careful examination of the particular case.

Cf. Chrystal's *Algebra*, xxvi. §§ 12 to 14.

12. *Power Series.*

It is beyond the scope of the present work to consider further the subject of infinite series in general; and we shall therefore confine our attention in what follows to series of the general form

$$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

as this form includes almost all the series which naturally arise in connexion with our subject.

In the above series we shall regard z as representing any complex number or vector, which may be replaced by $x + yi$, where x, y are scalars, or by ri^u, where r is the modulus of z or its tensor and therefore equal to the positive root $\sqrt{x^2 + y^2}$, and i^u its versor or u its argument measured in right angles. Coefficients a_0, a_1, a_2, . . . we shall regard generally as positive scalars, negative terms in the series being rendered positive by adding 2 (or $4\lambda + 2$) to the index u. In this way an alternating series, for instance, is the same function of $u + 2$ that the same series with all its terms positive is of u. In a few instances it may be desirable to regard the coefficients also as complex numbers or vectors: in such cases we shall write them explicitly as $a_0 i^{\alpha_0}$, $a_1 i^{\alpha_1}$, $a_2 i^{\alpha_2}$, . . .

The representative chain of links, when the coefficients are scalar, is (like the geometric series) one that is coiled so that each link makes the same angle with that preceding it, the lengths of the several links being those of a geometrical series altered in given fixed ratios according to the values of the coefficients

$a_0, a_1, a_2 \ldots$ If the coefficients are complex, the coil is further altered by bending at each angular point through the angles α_0, α_1, $\alpha_2 \ldots$

Writing the series in the form

$$a_0 + a_1 r i^u + a_2 r^2 i^{2u} + \ldots + a_n r^n i^{nu} + \ldots$$

it is obvious that, if the series obtained by putting $u = 0$, (so that it becomes scalar)

$$a_0 + a_1 r + a_2 r^2 + \ldots + a_n r^n + \ldots$$

is convergent, still more is the general series convergent. If the chain stretched out to its full length, (that is, so that the links are all in the same line,) has a finite length, any coiling or folding must bring its ends nearer together.

Hence, if the scalar series, which we may call the r-series, is convergent for all values of r, so is also the general series, which we may call the (r, u) series.

More commonly however the r-series is convergent only when r is less (or not greater) than some fixed quantity k; then the (r,u) series is certainly convergent when r is less (or not greater) than k, but it may be that when $r = k$, the (r, u) series is convergent, though the r-series is not so. We shall also show (what is not in itself obvious) that it is divergent, when r is greater than k.

It would seem that this last assertion is not necessarily true, when the coefficients are complex. Though the *regular* coiling (due to the versors, i^u, i^{2u}, &c.) may not bring the ends of the infinite chain to a finite distance from one another, it is conceivable that some farther folding (due to the versors i^{α_1}, $i^{\alpha_2} \ldots$) might do so.

A circle, whose centre is the origin and radius k, includes the extremities of all vectors from the origin whose tensors are less than k: this circle is termed the *circle of convergence* for the given series, and its radius k the *radius of convergence.*

13. *Continuity of the Power Series.*

For all points within the circle of convergence, that is, for all values of z, for which $\text{Mod}(z)$ or r is less than k, the series is a continuous function.

For, take a value z' near to z and also in the circle of convergence, then the rates of convergence of the series for both z and z' are finite, (since, if convergent for any value of r, the series must be still more convergent for a less value of r, the terms in the latter case having all less moduli than in the former), and therefore, when z' ultimately coincides with z, the sums of the two series become equal, by the reasoning in § 10. If, however, $r=k$, so that the extremity of z is on the circumference of the circle of convergence, the rate of convergence of the series for the value z' may become *infinitely slow*, as z' approaches without limit to z, and then, as we have seen, the limit of the sum of the series for the value z' is not necessarily the same as the sum for the value z.

14. *Convergency or Divergency of the Power Series.*

Taking the series in the (r,u) form, and denoting the remainder of the series after the first n terms by R_n, we have

$$R_n = a_n r^n i^{nu} + a_{n+1} r^{n+1} i^{\overline{n+1}u} + \ . \ . \ .$$
$$= a^n r^n i^{nu} \left\{ 1 + \frac{a_{n+1}}{a_n} r i^u + \frac{a_{n+2}}{a_n} r^2 i^{2u} + \ . \ . \ . \right\}$$

and since the sum of a finite number of terms is finite, the sum of the first n terms is finite, and it remains to consider under what conditions R_n is finite or infinite, or whether, though finite, it becomes indeterminate.

Let $\frac{a_n}{a_{n+1}} = k_n$, and let the limit, when n is infinite, of $\frac{a_n}{a_{n+1}}$, which we shall denote as $\llcorner \frac{a_n}{a_{n+1}}$, be k. Then if n be taken sufficiently great, though still finite, the series of fractions $\frac{a_{n+1}}{a_n}$, $\frac{a_{n+2}}{a_{n+1}}$, $\frac{a_{n+3}}{a_{n+2}}$, . . . will be successively nearer and nearer $1/k$ and by continuing the series sufficiently far a fraction will be obtained which differs from $1/k$ by as small a quantity as we please. Hence when n is sufficiently great but finite,

either $$\frac{1}{k_n} > \frac{a_{n+2}}{a_{n+1}} > \frac{a_{n+3}}{a_{n+2}} > \frac{a_{n+4}}{a_{n+3}} > \ . \ . \ . > \frac{1}{k}$$

or $$\frac{1}{k_n} < \frac{a_{n+2}}{a_{n+1}} < \frac{a_{n+3}}{a_{n+2}} < \frac{a_{n+4}}{a_{n+3}} < \ . \ . < \frac{1}{k}$$

In either case, since $\frac{a_{n+2}}{a_n} = \frac{a_{n+2}}{a_{n+1}} \cdot \frac{a_{n+1}}{a_n}$, $\frac{a_{n+2}}{a_n}$ is intermediate between $\frac{1}{k_n{}^2}$ and $\frac{1}{k^2}$, also $\frac{a_{n+3}}{a_n} = \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} \cdot \frac{a_{n+1}}{a_n}$, and therefore $\frac{a_{n+3}}{a_n}$ is intermediate between $\frac{1}{k_n{}^3}$ and $\frac{1}{k^3}$, and so on. Hence the series

$$1 + \frac{a_{n+1}}{a_n} ri^u + \frac{a_{n+2}}{a_n} r^2 i^{2u} + \ . \ . \ .$$

will be comprised between the two geometric series

$$1 + \frac{r}{k} i^u + \frac{r^2}{k^2} i^{2u} + \ . \ . \ .$$

and $$1 + \frac{r}{k_n} i^u + \frac{r^2}{k_n{}^2} i^{2u} + \ . \ . \ .$$

But, by taking n large enough, k_n can be made as near as we please to k, and therefore the latter series as near as we please to the former; hence the character of the series-factor of R_n as to convergency or divergency is determined by that of

$$1 + \frac{r}{k} i^u + \frac{r^2}{k^2} i^{2u} + \ . \ . \ .$$

We have seen that this series is convergent, if $\frac{r}{k} < 1$ and divergent, if $\frac{r}{k} > 1$, and therefore, the factor $a_n r^n i^{nu}$ being finite and determinate, R_n is finite if $r < k$, and infinite if $r > k$.

Hence the infinite series $a_0 + a_1 r i^u + a_2 r^2 i^{2u} + \dots$

is convergent, if $r < \llcorner \dfrac{a_n}{a_{n+1}}$, and divergent if $r > \llcorner \dfrac{a_n}{a_{n+1}}$.

In other words,

the series $a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$ *or* $a_0 + a_1 r i^u + \dots + a_n r^n i^{nu} + \dots$ *is convergent or divergent, according as* $r <$ *or* $>$ $\llcorner \dfrac{a_n}{a_{n+1}}$ *; that is, according as the extremity of the vector* z *drawn from the origin lies within or without the circle, whose centre is the origin and radius* $= k$ *or* $\llcorner \dfrac{a_n}{a_{n+1}}$, *termed the circle of convergence.*

Of course, if $k = 0$, the series is divergent for all values of z; while if $k = \infty$, it is convergent for all values of z.

The remaining case, where $r = k$, that is, mod. $(z) =$ radius of convergence or the extremity of the vector z lies on the circumference of the circle of convergence, requires further examination.

15. *Case where the tensor = radius of convergence, or* $r = k$.

If $r = k$ or $\llcorner \dfrac{a_n}{a_{n+1}}$, the limiting geometric series of the series-factor of R_n become

$$1 + i^u + i^{2u} + i^{3u} + \dots$$

and
$$1 + \frac{k}{k_n} i^u + \frac{k^2}{k_n^{\,2}} i^{2u} + \frac{k^3}{k_n^{\,3}} i^{3u} + \dots$$

the former of which is, as we have seen, a *fluctuating series* the radius of whose circle of fluctuation is $\frac{1}{2}$ cosec $\frac{u}{2}$, and which in the limiting case of $u = 0$ becomes divergent; while the latter is convergent or divergent as k_n is greater or less than k, that is, as the successive terms of $\dfrac{a_n}{a_{n+1}}, \dfrac{a_{n+1}}{a_{n+2}}, \dots$ continually approach without limit to k by decreasing or by increasing.

Also we have $R_n = a_n k^n i^{nu} \left\{ 1 + \frac{a_{n+1}}{a_n} \,.\, k i^u + \frac{a_{n+2}}{a_n} k^2 i^{2u} + \ \dots \right\}$ First, suppose $\llcorner(a_n k^n)$ to be infinite, then R_n is also infinite and the series is necessarily divergent.

Next, let $\llcorner(a_n k^n)$ be finite, then R_n will be finite or infinite as its series-factor is finite or infinite. Now if $k_n > k$, the series-factor of R_n is comprised between a *fluctuating* and a *convergent* series, and therefore cannot be *divergent*, but it may be *fluctuating*. On the other hand if $k_n < k$, the series-factor of R_n is comprised between a *fluctuating* and a *divergent* series, and therefore cannot be *convergent*, though it may be *fluctuating*. Whether R_n is actually fluctuating or not requires examination in each particular case. If it is fluctuating,

$$\text{the ultimate radius of fluctuation} = \llcorner(a_n k^n) \frac{1}{2} \operatorname{cosec} \frac{u}{2},$$

and the magnitude of this radius may be taken as a measure of the deviation of the series from convergency.

Lastly, let $\llcorner(a_n k^n) = 0$, then if $k_n > k$, so that the series-factor of R_n is either convergent or fluctuating, R_n ultimately vanishes, when $x = \infty$, and the series is convergent, the radius of fluctuation diminishing as n increases and ultimately vanishing. If however $k_n < k$, the expression for R_n, in the case where its series-factor is divergent, takes the form $0 \times \infty$ and no general conclusion can be drawn.

The foregoing reasoning supposes u to have a *finite* value different from 0. In the extreme case, where $u = 0$, the radius of fluctuation becomes infinite, the circle of fluctuation a straight line and the fluctuating series divergent, and the series itself the scalar series of ordinary algebra with all its terms positive.

In this case, unless $\llcorner a_n k^n = 0$, the series must be divergent, as is obvious, since the series becomes then

$$a_0 + a_1 k + a_2 k^2 + \ \dots \ + a_n k^n + \ \dots$$

Whether, when $\llcorner a_n k^n = 0$, the series is actually convergent or divergent must be determined by such tests of a simple character as may be found applicable to the particular case, which are given in the usual text-books of algebra, or by the more refined tests of Cauchy, De Morgan, Bertrand, &c. already referred to.

16. *Examples.*

The foregoing discussion will be elucidated by a careful study of the following examples:

(1) $1+z+\lfloor\underline{2}\,.\,z^2+\lfloor\underline{3}\,z^3+\ldots+\lfloor\underline{n}z^n+\ldots$ or $\Sigma_1^\infty\,(\lfloor\underline{n}z^n)$.

Here $k=\mathrm{L}\,\dfrac{\lfloor\underline{n}}{\lfloor\underline{n+1}}=\mathrm{L}\,\dfrac{1}{n+1}=0$, so that the circle of convergence is reduced to a point: the series therefore is divergent for all values of z, unless we may say that when $z=0$, its sum $=$ the first term $=1$,

(2) $1+z+\dfrac{z^2}{\lfloor\underline{2}}+\dfrac{z^3}{\lfloor\underline{3}}+\ \ldots$ or $\Sigma_1^\infty\left(\dfrac{z^n}{\lfloor\underline{n}}\right)$.

Here $k=\mathrm{L}\,\dfrac{\lfloor\underline{n+1}}{\lfloor\underline{n}}=\mathrm{L}\,(n+1)=\infty$, so that for every finite value of x, $\dfrac{r}{k}=0$, and therefore the series is convergent for every finite value of z.

(3) $1+2z+3z^2+\ldots+(n+1)\,z^2+\ldots$ or $\Sigma\,(\overline{n+1}x^n)$.

Here $k=\mathrm{L}\,\dfrac{n+2}{n+1}=1$, or the radius of convergence is 1, so that, if $r<1$, the series is convergent and if $r>1$, divergent. If $r=1$, the series is $1+2i^u+3i^{2u}+\ \ldots$, which is obviously divergent, as also appears from the fact that $\mathrm{L}\,(a_nk^n)=\mathrm{L}\,(n+1)=\infty$.

(4) $z+\dfrac{z^2}{2}+\dfrac{z^3}{3}+\ldots$ or $\Sigma_1^\infty\left(\dfrac{z^n}{n}\right)$.

Here $k=\mathrm{L}\,\dfrac{n+1}{n}=1$, and the series is convergent or divergent, as r is less or greater than 1.

If $r = 1$, the series $i^u + \frac{1}{2} i^{2u} + \frac{1}{3} i^{3u} + \dots$, and since

$$\llcorner a_n k^n = \llcorner \frac{1}{n} = 0,$$

the series is convergent: except only, when $u = 0$, in which case the series is $1 + \frac{1}{2} + \frac{1}{3} + \dots$, which as we have shown above (§ 8) is divergent.

(5) $1 + \frac{2}{1} z + \frac{3}{2} . z^2 + \dots$ or $\Sigma_1^{\infty} \left(\frac{n+1}{n} z^n \right)$.

Here $k = \llcorner \frac{n+1}{n} \Big/ \frac{n+2}{n+1} = \llcorner \frac{(n+1)^2}{n(n+2)} = 1$, so that the series is convergent, if $r < 1$, divergent if $r > 1$. If $r = 1$, the series is

$$1 + \tfrac{2}{1} i^u + \tfrac{3}{2} i^{2u} + \tfrac{4}{3} i^{3u} + \quad \dots,$$

$$k_n = \frac{n^2 + 2n + 1}{n^2 + 2n} > 1 \text{ or } k, \text{ and } \llcorner (a_n k^n) = \llcorner \frac{n+1}{n} = 1,$$

therefore the series cannot be divergent. It is in fact fluctuating, for it may be put into the form

$$(1 + i^u + i^{2u} + i^{3u} + \dots) + (i^u + \tfrac{1}{2} i^{2u} + \tfrac{1}{3} i^{3u} + \dots),$$

that is, a fluctuating series + a convergent series or finite quantity: and the radius of fluctuation $= \frac{1}{2} \operatorname{cosec} \frac{u}{2}$.

If $u = 0$, the radius of fluctuation is infinite, and the series, which then becomes $1 + 2 + \frac{3}{2} + \frac{4}{3} + \dots$, is divergent, as is evident. If $u = -2$, the series is $1 - 2 + \frac{3}{2} - \frac{4}{3} + \dots$, which being put in the form $(1 - 1 + 1 - 1 + \dots) - (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots)$ is obviously an oscillating series, the limits of the oscillation being $-\log_e 2$ and $1 - \log_e 2$, since (as will be proved) $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

(6) $1 + \frac{1}{2} z + \frac{2}{3} z^2 + \dots + \frac{n}{n+1} z^n + \dots$ or $\Sigma_1^{\infty} \left(\frac{n}{n+1} z^n \right)$.

Here, as in the last example, $k = 1$ and the series is convergent if $r < 1$, divergent if $r > 1$.

If $r = 1$, $k^n = \dfrac{n\,(n+2)}{(n+1)^2} < 1$ or k, and $\llcorner\,(a_n k^n) = \llcorner\,\dfrac{n}{n+1} = 1$, therefore the series cannot be convergent. It is in fact fluctuating, for it may be expressed as

$$1 + i^u + i^{2u} + \dots - \left(\tfrac{1}{2}\, i^u + \tfrac{1}{3}\, i^{2u} + \tfrac{1}{4}\, i^{3u} + \dots\right),$$

of which the first member is fluctuating and the second convergent.

If $u = 0$, the series is $1 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$, which, being greater than $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$, is obviously divergent.

If $u = -2$, the series is $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \dots$ and its sum oscillates between $1 - \log_e 2$ and $2 - \log_e 2$.

(7) $\Sigma_0^\infty \dfrac{z^n}{a^n + b}$ or $\dfrac{1}{1+b} + \dfrac{z}{a+b} + \dfrac{z^2}{a^2+b} + \dfrac{z^3}{a^3+b} + \dots$

Here $k = \llcorner\,\dfrac{a^{n+1}+b}{a^n+b} = a$, if $a > 1$, but $= 1$, if $a < 1$ or $= 1$.

a. If $a > 1$, the radius of convergence being $= a$, the series is convergent if $r < a$, divergent if $r > a$.

If $r = a$, $\llcorner\, a_n k^n = \llcorner\,\dfrac{a^n}{a^n+b} = 1$, and $k_n = \dfrac{a^{n+1}+b}{a^n+b} > 1$ so that $\dfrac{k}{k_n} < 1$: hence the series cannot be divergent. The series being in this case $\dfrac{1}{1+b} + \dfrac{a i^u}{a+b} + \dfrac{a^2 i^{2u}}{a^2+b} + \dots$ may be put into the form

$$(1 + i^u + i^{2u} + \dots) - b\left(\frac{1}{1+b} + \frac{1}{a+b}\, i^u + \frac{1}{a^2+b}\, i^{2u} + \dots\right)$$

of which the first member is fluctuating (radius of fluctuation $= \frac{1}{2}\operatorname{cosec}\dfrac{u}{2}$) and the second is convergent, since it is the series itself when $r = 1$, which is less than the radius of convergence, a :

so that when $r=a$, the series is fluctuating. In the limiting case of $u=0$, the series is

$$\frac{1}{1+b}+\frac{a}{a+b}+\frac{a^2}{a^2+b}+\ldots$$

and is divergent.

β. If $a=1$, the series reduces to a geometric series, and is therefore convergent, fluctuating or divergent, as $r <, =,$ or > 1.

γ. If $a<1$, $k=\llcorner\,\dfrac{a^{n+1}+b}{a^n+b}=1$, and the series is convergent if $r<1$, divergent if $r>1$.

If $r=1$, $\llcorner\, a_n k^n=\llcorner\,\dfrac{1}{a^n+b}=\dfrac{1}{b}$, and $k_n=\dfrac{a^{n+1}+b}{a^n+b}<1$ so that $\dfrac{k}{k_n}>1$: hence the series cannot be convergent. In this case the series is

$$\frac{1}{1+b}+\frac{i^u}{a+b}+\frac{i^{2u}}{a^2+b}+\ldots \text{ or } \frac{1}{b}\left\{(1+i^u+i^{2u}+\ldots)\right.$$
$$\left.-\left(\frac{1}{1+b}+\frac{ai^u}{a+b}+\frac{a^2i^{2u}}{a^2+b}+\ldots\right)\right\}$$

in which latter form the second member is convergent, since $a<1$ and the first fluctuating. Hence the series is fluctuating, and the ultimate radius of fluctuation is $\dfrac{1}{2b}\operatorname{cosec}\dfrac{u}{2}$, which becomes infinite when $u=0$.

EXAMPLES.

1. Using the figure and notation of §§ 2 and 4,

(1) Prove that $\overline{RP} = \frac{r}{1-r^2}\, i^{u+2v}$

$$\text{and } \overline{MP} = \frac{1}{2}\frac{\overline{OP_2}}{\overline{OP_2'}} = \frac{1}{2}\sqrt{\frac{1+2r\cos u + r^2}{1-2r\cos u + r}}\,.\,i^w,$$

where $\tan v = \dfrac{r\sin u}{1-r\cos u}$ and $\tan w = \dfrac{2r\sin u}{1-r^2}$.

(2) If $S = \rho i^v$, prove and illustrate by diagrams that

a. $ri^u + \rho^{-1} i^{-v} = 1.$

β. $\rho^{-2} = 1 - 2r\cos u + u^2$ and $r^2 = 1 - 2\rho^{-1}\cos v + \rho^{-2}.$

$$\gamma.\quad \left.\begin{aligned} \frac{1}{r}\cos u &= 1 + \frac{1}{\rho}\cos v + \frac{1}{\rho^2}\cos 2v + \dots \\ \frac{1}{r}\sin u &= \frac{1}{\rho}\sin v + \frac{1}{\rho^2}\sin 2v + \dots \end{aligned}\right\}$$

provided $r < 1$ and $\cos u > \dfrac{r}{2}$.

$$\delta.\quad \frac{1}{2}\cos u \sec u + \frac{1}{2^2}\cos 2u \sec^2 u + \frac{1}{2^3}\cos 3u \sec^3 u + \dots = \cos 2u.$$

$$\frac{1}{2}\sin u \sec u + \frac{1}{2^2}\sin 2u \sec^2 u + \frac{1}{2^3}\sin 3u \sec^3 u + \dots = \sin 2u.$$

(3) Shew that O, P_1, P_2 . . . are points on an equiangular spiral, whose focus is P.

(4) Determine P by a construction from the expressions for the project and traject of $\overline{OP}$.

(5) Given the position of P determine the series by geometrical construction.

2. Prove that, z denoting the complex number or vector ri^u, the radii of convergence of the several series $\Sigma_1^\infty (na_n z^n)$, $\Sigma_1^\infty \left(\dfrac{a_n z^n}{n}\right)$, $\Sigma_1^\infty (n^a a_n z^n)$ are equal to that (k) of the series $\Sigma_1^\infty (a_n z^n)$.

Also that, if $\Sigma_1^\infty (a_n z_n)$ is convergent when $r=k$, so also is $\Sigma_1^\infty \left(\frac{a_n}{n} z^n\right)$, while $\Sigma_1^\infty (n a_n z^n)$ may be fluctuating or divergent: and that, if $\Sigma_1^\infty (a_n z^n)$ is divergent, so also is $\Sigma_1^\infty (n a_n z^n)$, while $\Sigma_1^\infty \left(\frac{a_n}{n} z^n\right)$ may be fluctuating or convergent.

3. For the following series find k (the radius of convergence): and determine, when $r=k$, if the series is convergent, fluctuating or divergent. In the case of fluctuation, determine the radius of fluctuation. Examine specially the limiting case, when $u=0$ or the series becomes scalar.

(1) $1 + 1 \,.\, 2z + 2 \,.\, 3z^2 + \ldots + n(n+1) z^n + \ldots$

(2) $1 + \frac{1}{2} z + \frac{2}{3} z^2 + \ldots + \frac{n}{n+1} z^n + \ldots$

(3) $1 + \frac{1^2}{2} z + \frac{2^2}{3} z^2 + \ldots + \frac{n^2}{n+1} z^n + \ldots$

(4) $1 + \frac{1}{2^2} z + \frac{2}{3^2} z^2 + \ldots + \frac{n}{(n+1)^2} z^n + \ldots$

(5) $1 + 3z + 6 \,.\, \frac{z^2}{2^2} + \ldots + \frac{(n+1)(n+2)}{1 \,.\, 2} \frac{z^n}{n^2} + \ldots$

(6) $\Sigma_1^\infty \left(\frac{n^2-1}{n^2+1} \,.\, z^n\right)$, (7) $\Sigma_1^\infty \left(\frac{n+1}{n^p} z^n\right)$, (8) $\Sigma_1^\infty \{(2^{n+1}+1) z^n\}$,

(9) $\Sigma_1^\infty \{n(3^n-1) z^n\}$, (10) $\Sigma_1^\infty \left(\frac{3^n+1}{n} z^n\right)$,

(11) $1 + \frac{1}{2} z + \frac{1 \,.\, 3}{2 \,.\, 4} z^2 + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \,.\, z^3 + \ldots$

$\left(\text{For } \llcorner \frac{1 \,.\, 3 \,.\, 5 \,.\, 7 \ldots}{2 \,.\, 4 \,.\, 6 \,.\, 8 \ldots}, \text{ see Ch. X. § 10}\right)$,

(12) $1 + \frac{1}{2} z + \frac{1 \,.\, 3}{2 \,.\, 4} \,.\, 2z^2 + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \,.\, 3z^3 + \ldots$

(13) $1 + \frac{1}{2} z + \frac{1 \,.\, 3}{2 \,.\, 4} \sqrt{2} \,.\, z^2 + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \,.\, \sqrt{3} \,.\, z^3 + \ldots$

(14) $1 + \frac{1}{2} z + \frac{1 \,.\, 3}{2 \,.\, 4} \frac{z^2}{2} + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \frac{z^3}{3} + \dots$

(15) $\Sigma_1^\infty \left(\frac{z^n}{a^n - 1} \right)$. (16) $\Sigma_1^\infty \left(n^{\frac{1}{2}} z^{n-1} \right)$. (17) $\Sigma_1^\infty \left\{ \frac{(\underline{|n})^2}{\underline{|2n}} z^{n-1} \right\}$.

(18) $\Sigma_1^\infty \left\{ \frac{\underline{|2n}}{(\underline{|n})^2} z^{n-1} \right\}$.

(19) $\Sigma_1^\infty \left\{ \frac{n^n}{\underline{|n}} z^n \right\}$.

(See Hall and Knight's *Higher Algebra*, p. 246, ed. 1887.)

(20) $$1 + \frac{\alpha \,.\, \beta}{1 \,.\, \gamma} z + \frac{\alpha \,.\, \overline{\alpha+1} \,.\, \beta \,.\, \overline{\beta+1}}{1 \,.\, 2 \,.\, \gamma \overline{\gamma+1}} z^2$$
$$+ \frac{\alpha \,.\, \overline{\alpha+1} \,.\, \overline{\alpha+2} \,.\, \beta \,.\, \overline{\beta+1} \,.\, \overline{\beta+2}}{1 \,.\, 2 \,.\, 3 \,.\, \gamma \,.\, \overline{\gamma+1} \, \overline{\gamma+2}} z^3 + \dots$$

CHAPTER IX

EXPANSIONS AND SUMMATIONS

In the ordinary treatises on *Elementary Algebra*, the developments of a few finite expressions into infinite series of terms are established for scalar values of the symbols: in the present chapter the extension of these to vector values or complex numbers, and some important results thence deducible will be considered.

It will be convenient to premise the following general method of obtaining Trigonometrical Series from any given Algebraical Expansion.

1. *Trigonometrical Series deducible from Algebraical Expansions.*

Let ϕz denote any function, which can be developed by algebraical processes into the series (finite or infinite in the number of its terms)

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

Then ϕz, having been proved (it may be) to be the generating function of this series with some restriction on the meanings of some or all of the general symbols involved (as, that they are scalar, or positive, or integral), must by the principle of the "Permanence of Equivalent Forms" (see Introd. p. 4) be also the generating function of the same series universally; and understanding the sign of equality in this sense, we may assert for all values of z, that

$$\phi z = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

It must always be borne in mind however that, in the case of the series consisting of an infinite number of terms, *quantitative equality* can only be asserted when the infinite series has been shown to be *convergent.*

Let z be a vector or complex number, denoted by $x+iy$ or ri^u or $r\epsilon^{i\theta}$, where x, y are scalars, r, *a positive* scalar, u, θ, the measures in right angles and circular measure respectively of the inclination of z to the prime vector,

$$\begin{aligned}\text{then } \phi z = \phi(ri^u) &= a_0 + a_1 ri^u + a_2 r^2 i^{2u} + \ldots + a_n r^n i^{nu} + \ldots \\ &= a_0 + a_1 r(\cos u + i \sin u) + \ldots + a_n r^n (\cos nu + i \sin nu) + \ldots \\ &= a_0 + a_1 r \cos u + a_2 r^2 \cos 2u + \ldots + a_n r^n \cos nu + \ldots \\ &\quad + i(a_1 r \sin u + a_2 r^2 \sin 2u + \ldots + a_n r^n \sin nu + \ldots)\end{aligned}$$

Let ϕz or $\phi(x+iy)$ be reduced to the form $U+iV$, where U, V are scalars, then, equating projects and trajects of the vector ϕz,

$$\begin{aligned}U &= a_0 + a_1 r \cos u + a_2 r^2 \cos 2u + \ldots + a_n r^n \cos nu + \ldots \\ V &= a_1 r \sin u + a_2 r^2 \sin 2u + \ldots + a_n r^n \sin nu + \ldots\end{aligned}$$

The quantities U, V are termed *conjugate* functions, and the two equivalent series *conjugate* series.

This result may be used, as will be shown in the sequel, either to obtain new trigonometrical series from a known algebraical expansion, or to sum certain given trigonometrical series by means of a known algebraical summation.

2. *Newton's Binomial Theorem.*

The theorem is that

$$(1+z)^m = 1 + mz + \frac{m \,.\, \overline{m-1}}{1\,.\,2} \,.\, z^2 + \ldots + \frac{m \,.\, \overline{m-1} \,..\, \overline{m-n+1}}{1\,.\,2 \ldots n} z^n + \ldots$$

The right-hand side of this equation has a finite number of terms if m is a positive integer, but in all other cases it is an infinite series.

It is proved, in the case of m a positive integer, by considering the product of the m factors, $1+z_1$, $1+z_2$, . . . , $1+z_m$, in which it is clear, from the process of multiplication, that the terms consisting of the product of n z's will be as many in number as there are different combinations of n out of the m letters $z_1, z_2 \ldots, z_m$. These terms will all reduce to z^n, if each of the quantities $z_1, z_2, \ldots, z_m$ is equal to z, and their sum will then be $\frac{m \,.\, \overline{m-1} \ldots \overline{m-n+1}}{1 \,.\, 2 \ldots n} z^n$, so that the series for $(1+z)^m$ will be that written above, terminating with the term z^m.

In the foregoing result, deduced directly from the fundamental laws of combination of algebraical symbols, there is no restriction whatever on the meaning of z; and the restriction that m is a positive integer may be removed by the consideration that, the *form* of the equation being perfectly general, and the meanings of indices in the successive cases of fractional, negative, or complex indices having been obtained in accordance with the same fundamental laws as were shown to hold for positive integers, the result must, by the Principle of the Permanence of Equivalent Forms,* be true without any restriction as to the value of m.

It is however to be specially observed that in passing from the case of m a positive integer to any other case the series changes from one of a finite number of terms to an infinite series. Hence, though the *formal* equality of the finite expression and the infinite series, regarding the latter as derived from the former as its generating function, will still hold, it cannot be asserted of the series regarded as the *quantitative* sum of its terms that the finite expression and the series are *quantitatively* equal, unless it is proved that the limit of the sum of the terms of the series is finite and definite, or, in other words, that the series is convergent. In fact (as has been shown in the preceding chapter § 11) it cannot be asserted of series, which are not absolutely convergent, that the fundamental Laws of Algebra hold without limitation.

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* It is usual to give a distinct proof of the theorem for fractional and negative indices, and the one most commonly adopted is that known as Euler's proof. An examination of this proof however will show that it does in fact assume the principle we have just used, though perhaps in a case in which it is easier for the beginner in Algebra to see its truth than when asserted as above in its complete generality.

3. *Convergency or Divergency of the Binomial Series.*

Confining ourselves to the case, where m is a scalar, the general term being

$$\frac{m.(m-1)\dots(m-n+1)}{1.2\dots n}z^n,$$

the ratio of the $\overline{n+1}^{\text{th}}$ term to the n^{th} is

$$\frac{m-n+1}{n}.z \text{ or } -\left(1-\frac{m+1}{n}\right)ri^u \text{ or } \left(1-\frac{m+1}{n}\right)ri^{u+2}.$$

In this last form the coefficient $\left(1-\dfrac{m+1}{n}\right)$ is positive for all values of n greater than $m+1$, and its limit, when $n=\infty$, is 1: hence, by the test established in the last chapter, the series is convergent if $r<1$, and divergent if $r>1$.

If $r=1$, the remainder after n terms is equal to the product of

$$\left(1-\frac{m+1}{1}\right)\left(1-\frac{m+1}{n}\right)\dots\left(1-\frac{m+1}{n}\right)i^{n(u+2)}$$

and a series, which ultimately, as n is increased, approximates without limit to the series

$$1+i^{u+2}+i^{2(u+2)}+\dots$$

which, except in the case where $u+2=4\lambda$, (λ being an integer) is a fluctuating series, whose radius of fluctuation

$$=\tfrac{1}{2}\times\operatorname{cosec}\frac{u+2}{2}=\tfrac{1}{2}\sec\frac{u}{2}.$$

Hence the remainder is ultimately a fluctuating series, whose radius of fluctuation is the limit of

$$\left(1-\frac{m+1}{1}\right)\left(1-\frac{m+1}{2}\right)\dots\left(1-\frac{m+1}{n}\right).\tfrac{1}{2}\sec\frac{u}{2}, \text{ when } n=\infty.$$

Now, since the series $(m+1)\left(\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}+\dots\right)$ is divergent, (cf. § 10, Ex. 4 in the last Chapter), the infinite product of factors* in the expression for the radius of fluctuation is 0, if $m+1$ is positive, and ∞ if $\overline{m+1}$ is negative. Hence in the case of $r=1$, the Binomial Series is convergent, if $m+1$ is positive, and divergent if $\overline{m+1}$ is negative; while, if $m+1=0$, it is truly fluctuating, since in this case the series reduces to the case of a Geometric Series, of which a unit-vector is the common ratio, discussed in the last Chapter.

In the exceptional case of $r=1$ and $u=4\lambda-2$ (λ integral), $z=i^{4\lambda-2}=-1$, so that $(1+z)^m=(1-1)^m=0$ or ∞, according as m is positive or negative.

The series in this case becomes

$$1-m+\frac{m.\overline{m-1}}{1.2}-\frac{m.\overline{m-1}\,\overline{m-2}}{1.2.3}+\dots$$

and we may expect to find that it converges to 0, if m is positive, and is divergent, if m is negative.

Taking the sum of the first $\overline{n+1}$ terms, it may readily be proved that

$$1-m+\frac{m.\overline{m-1}}{1.2}+\dots+(-1)^n\frac{m.\overline{m-1}\dots\overline{m-n+1}}{1.2\dots n}$$
$$=\left(1-\frac{m}{1}\right)\left(1-\frac{m}{2}\right)\left(1-\frac{m}{3}\right)\dots\left(1-\frac{m}{n}\right).$$

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* See C. Smith's *Algebra*, Chap. xxvi. or Chrystal's *Algebra*, Chap. xxvi §§ 22—27.

For if the product on the right-hand side of this identity is denoted by u_n,

$$u_n - u_{n-1} = \left(1-\frac{m}{1}\right)\left(1-\frac{m}{2}\right)\ldots\left(1-\frac{m}{n-1}\right)\left\{\overline{1-\frac{m}{n}}-1\right\}$$

$$=\frac{1-m}{1}\cdot\frac{2-m}{2}\ldots\frac{\overline{n-1}-m}{n-1}\cdot\frac{-m}{n}$$

$$=(-1)^n\frac{m\,.\,(m-1)\,(m-2)\ldots(m-n+1)}{1\,.\,2\,.\,3\ldots n}$$

but $u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + \ldots + (u_{n-1} - u_{n-2}) + (u_n - u_{n-1})$

$$\therefore u_n = 1 - m + \frac{m\,.\,\overline{m-1}}{1\,.\,2} - \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}}{1\,.\,2\,.\,3} + \ldots$$

$$+(-)^n\frac{m\,.\,\overline{m-1}\ldots(m-n+1)}{1\,.\,2\ldots n}.$$

Now the product u_n, when n is increased without limit, is (as we have seen above) ultimately 0 or ∞, according as m is positive or negative; therefore the infinite series above converges to 0, if m is positive, and is divergent if m is negative.

4. *Illustrative Diagrams.*

The preceding section may be well illustrated by a few diagrams drawn for particular cases, when $r = 1$ or z is a unit-vector.

1. Let $m = \frac{5}{2}$ or $-\frac{5}{2}$, then the series are

$$1+\frac{5}{2}i^u+\frac{5\,.\,3}{2\,.\,4}i^{2u}+\frac{5\,.\,3\,.\,1}{2\,.\,4\,.\,6}i^{3u}-\frac{5\,.\,3\,.\,1\times 1}{2\,.\,4\,.\,6\,.\,8}i^{4u}$$

$$+\frac{5\,.\,3\,.\,1\times 1\,.\,3}{2\,.\,4\,.\,6\,.\,8\,.\,10}i^{5u} - \&c.$$

$$1-\frac{5}{2}i^u+\frac{5\,.\,7}{2\,.\,4}i^{2u}-\frac{5\,.\,7\,.\,9}{2\,.\,4\,.\,6}i^{3u}+\frac{5\,.\,7\,.\,9\,.\,11}{2\,.\,4\,.\,6\,.\,8}i^{4u} - \&c.$$

The diagrams representing the first four terms of these series in the case where $u = \frac{2}{3}$, are figs. 1 and 2 respectively, the successive terms being $\overline{OA_1}$, $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, in each, and $\overline{OA_4}$ therefore their sum. The remainder after the first three terms of the first series will be a vector drawn from A_3 to some point within the circle of fluctuation passing through A_3, A_4 and divided by A_3A_4 into segments containing the angles $\frac{2^{\llcorner}}{3}$, $\frac{4^{\llcorner}}{3}$ respectively. The limit of the series is, by the Binomial Theorem, $(1 + i^{\frac{2}{3}})^{\frac{5}{2}}$, which reduces to $3^{\frac{5}{4}} i^{\frac{5}{6}}$ and is represented by $\overline{OZ}$, such that $OZ = 3{\cdot}93$ and $ZOA_1 = 75°$. It appears that Z lies within the circle through A_3, A_4, the radius of which

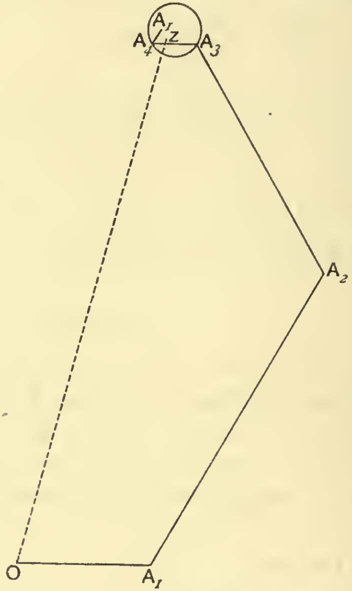


FIG. 1.

$$\text{(being equal to } \frac{5\,.\,3\,.\,1}{2\,.\,4\,.\,6} \cdot \frac{1}{2} \sec \frac{1}{3} = \frac{5}{48} \sqrt{3} = {\cdot}18 \text{ nearly)}$$

affords a sort of measure of the degree of approximation obtained by taking the first 3 or 4 terms. The convergency of the series is thus obvious.

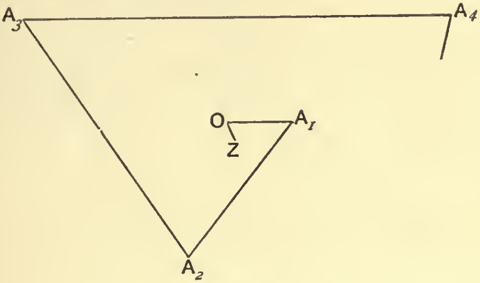


FIG. 2.

The generating function of the latter series becomes $(1+i^{\frac{2}{3}})^{-}$ or $3^{-\frac{5}{4}}i^{-\frac{5}{6}}$, and its divergency is evident.

(2) The cases represented in figs. 3, 4 are those of

$$(1-i^u)^{\frac{1}{2}}=1-\tfrac{1}{2}i^u-\frac{1\,.\,1}{2\,.\,4}\,.\,i^{2u}-\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}i^{3u}-\frac{1\,.\,1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}i^{4u}-\dots$$

$$(1-i^u)^{-\frac{1}{2}}=1+\tfrac{1}{2}i^u+\frac{1\,.\,3}{2\,.\,4}i^{2u}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}i^{3u}+\dots$$

taken for the value $u=\frac{2}{3}$. The values of OZ are $i^{-\frac{1}{3}}$, $i^{\frac{1}{3}}$ respectively. Observe the slowness of the convergence in fig. 4 as

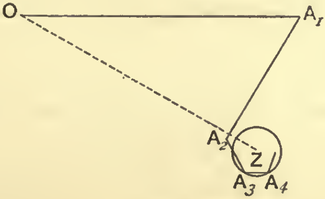


FIG. 3.

compared with fig. 3. As u diminishes, this slowness increases, until finally, when $u=0$, the second series becomes divergent.

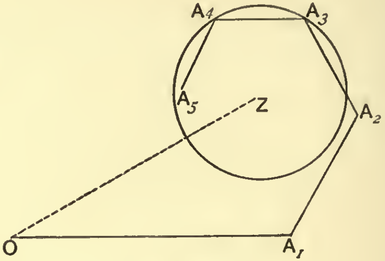


FIG. 4.

5. *The Binomial Series equal to the* prime *value of* $(1+z)^m$.

The expression $(1+z)^m$ is, as we have seen (p. 45), a function which has in general an infinite number of values, which however are all coincident, if m is an integer, and reduce to a finite number of distinct values, when the index is scalar, and a finite fraction. The Binomial Series however is a single-valued function. It is necessary therefore to determine to which of the values of $(1+z)^m$ the series must be taken to be equal. It will be sufficient here to confine our investigation to the case, where m is a scalar.

Let $1+z=\rho i^v$, then $\rho i^v=1+ri^u=1+r\cos u+ir\sin u$, so that

$$\rho\cos v=1+r\cos u \text{ and } \rho\sin v=r\sin u,$$

whence

$$\rho=\left\{1+2r\cos u+r^2\right\}^{\frac{1}{2}} \text{ and } \tan v=\frac{r\sin u}{1+r\cos u}.$$

It follows that

$$\frac{\sin u}{\rho}=\frac{\sin v}{r}=\frac{\sin u\cos v-\cos u\sin v}{\rho\cos v-r\cos u}=\frac{\sin(u-v)}{1},$$

and if r be not greater than 1 (as must be the case for convergency), since $\sin(u-v)$ lies between -1 and $+1$, $\sin v$ is limited to lie between $-r$ and r. Also ρ and $1+r\cos u$ being both positive, $\cos v$ is positive, so that the value of v (within the limits $-2^{\llcorner}$ and $+2^{\llcorner}$) is a positive or negative acute angle, whose sine is numerically not greater than r.

If this angle is denoted by v_0, the general value of v is $v_0+4\lambda$, (λ integral), hence the general value of $(1+z)^m$ is $\rho^m i^{mv_0+4\lambda m}$. If m is integral, this has only one distinct value; but if m is a finite fraction $=\frac{p}{q}$, where p, q are integers prime to each other, it has q distinct values corresponding to the values $0, 1, 2 \ldots q-1$ of λ: while, if m is incommensurable, it has an infinite number of distinct values for each different value of λ.

Now if $u=0$ or a multiple of $4^{\llcorner}$, $v_0=0$, and both z and the binomial series are scalar; but $\rho^m i^{4\lambda m}$, when m is not integral, is scalar only if $\lambda=0$, if m is incommensurable, or if $\lambda=0$ or some multiple of q, if $m=\frac{p}{q}$. Hence, if $u=0$, λ must be taken as 0: and as v changes continuously with u, λ must also be taken as 0, when u has any value.

The annexed diagram will throw light on the foregoing discussion. In this $\overline{OI}$ is the prime vector, $\overline{IP}=z=ri^u$, so that $\overline{OP}=1+z$, and to represent the case of a convergent series for which $r<1$, $IP<OI$. Then $OP=\rho$ and the angle $IOP=v_0{}^{\llcorner}$, and since P is some point on the circle, radius r and centre I, or APA', the limits of ρ are OA, OA' or $1+r$ and $1-r$, while if OB, OB' are tangents to the circle APA' drawn from O, OP lies between OB' and OB, and the limits of v_0 are the positive and negative angles BOI, $B'OI$, whose sine (BI) is equal to r. Hence $\overline{1+z}^m$ or $\overline{OP}^m$ will be one of the vectors $\overline{OQ_0}$, $\overline{OQ_1}$, $\overline{OQ_2}\ldots$ whose tensors are each ρ^m or OP^m and inclinations $Q_0OI=m\,.\,POI$, $Q_1OI=m\,.\,POI+4m^{\llcorner}$, $Q_2OI=m\,.\,POI+8m^{\llcorner}\ldots$ respectively. (The diagram is drawn for $m=\frac{2}{3}$.) But it is clear from the figure that the limit of the sum of the vectors in the Binomial Series,

$\overline{OI}+m\,.\,\overline{IP}+\frac{m\,.\,\overline{m-1}}{1\,.\,2}\overline{IP}^2+\ldots$ is the vector $\overline{OQ_0}$ and not $\overline{OQ_1}$, or $\overline{OQ_2}\ldots$, so that the series is equal to $\rho^m\,i^{mv_0}$, v_0 being the acute angle POI.

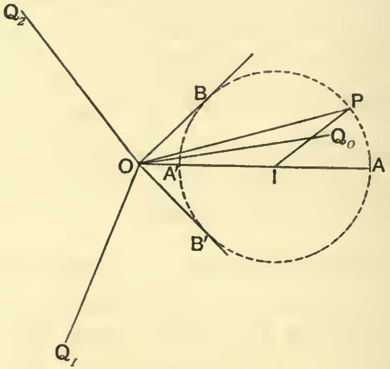


FIG. 5.

Ex. If $m<1$, the locus of Q_0 corresponding to the circular locus of P, when r is given, is an oval lying within the circle Examine the nature of the locus for values of $m>1$.

6. *General Form of the Binomial Theorem.*

The foregoing results may be thus summarized.

If $z=ri^u$, and $1+z=\rho i^{v_0}$, where $\rho=\{1+2r\cos u+r^2\}^{\frac{1}{2}}$, and v_0 is a positive or negative acute angle, such that

$$\tan v_0=\frac{r\sin u}{1+r\cos u},$$

the Binomial Theorem that

$$(1+z)^m=1+mz+\frac{m\,.\,\overline{m-1}}{1\,.\,2}z^2+\ldots$$

$$+\frac{m\,.\,\overline{m-1}\ldots\overline{m-n+1}}{1\,.\,2\ldots n}z^n+\ldots$$

or

$$\rho^m i^{mv_0} = 1 + mri^u + \frac{m\,.\,\overline{m-1}}{1\,.\,2}\, r^2 i^{2u} + \dots$$
$$+ \frac{m\,.\,\overline{m-1}\dots\overline{m-n+1}}{1\,.\,2\dots n}\, r^n i^{nu} + \dots$$

is *quantitatively* true (the infinite series being convergent) if $r < 1$: but not *quantitatively* true (the infinite series being divergent) if $r > 1$, unless m be a positive integer, in which case the number of terms of the series is finite. If $r = 1$, the theorem is quantitatively true, provided $m > -1$; but not so, if $m = -1$ or $m < -1$, for then in the former case the series is fluctuating and in the latter divergent: with the single exception that, if $u = 4\lambda + 2$ (λ integral), it is quantitatively true, if $m > 0$, but not so, the series being divergent, if m is negative.

If the index m is a (vector or) complex number, it may be shown, just as for scalar values of m, that the series is convergent or divergent, and therefore quantitatively equal to the prime value of $(1 + z)^m$ or not, according as r is less or greater than 1. The case, where $r = 1$, requires farther examination.

Denoting the successive coefficients of the powers of z by $1_m, 2_m, 3_m \dots$, we have to inquire under what conditions the series

$$1 + 1_m\,.\,i^u + 2_m\,.\,i^{2u} + 3_m\,.\,i^{3u} + \dots$$

is quantitatively equal to $(1 + i^u)^m$, when $m = \mu + i\nu$, where μ and ν are scalars.

The coefficient of i^{nu} or $n_m = \dfrac{m\,.\,\overline{m-1}\dots\overline{m-n+1}}{1\,.\,2\dots n}$ may be conveniently written in the shape

$$(-1)^n\,.\left(1 - \frac{m+1}{1}\right)\left(1 - \frac{m+1}{2}\right)\dots\left(1 - \frac{m+1}{n}\right).$$

Let the successive factors of this product be $k_1 i^{\lambda_1}, k_2 i^{\lambda_2}, \dots k_n i^{\lambda_n}$, so that

$$k_n\, i^{\lambda_n} = \left(1 - \frac{\mu + 1}{n} - i\,\frac{\nu}{n}\right),$$

whence

$$k_n = \left(1 - 2\,\frac{\mu + 1}{n} + \frac{\overline{\mu + 1}^2 + \nu^2}{n^2}\right)^{\frac{1}{2}}$$

and

$$\tan \lambda_n = -\frac{\nu}{n-\mu-1} :$$

then

$$n_m = (-1)^n \, . \, k_1 \, k_2 \, . \, . \, . \, k_n \, i^{\lambda_1+\lambda_2+\ldots+\lambda_n}.$$

Now, when p is large, λ_p is small and $k_p = 1 - \frac{\mu+1}{p}$ very nearly neglecting the terms in its expansion containing powers of $\frac{1}{p}$ higher than the first, so that after some large but finite value p, we may write

$$k_p \, k_{p+1} \, . \, . \, . \, k_n = \left(1 - \frac{\mu+1}{p}\right)\left(1 - \frac{\mu+1}{p+1}\right) \, . \, . \, . \left(1 - \frac{\mu+1}{n}\right).$$

But the series

$$\frac{1}{p} + \frac{1}{p+1} + \frac{1}{p+2} + \, . \, . \, . + \frac{1}{n}$$

increases without limit with n, hence the product $k_p \, k_{p+1} \ldots k_n$ is infinite, when n is increased without limit, if $\mu + 1$ is negative, and 0, if $\mu + 1$ is positive : therefore the series is convergent if $\mu > -1$ and divergent if $\mu < -1$.

In the case where $\mu = -1$, $k_n = \left(1 + \frac{\mu^2}{n^2}\right)^{\frac{1}{2}}$, and the limit of the product

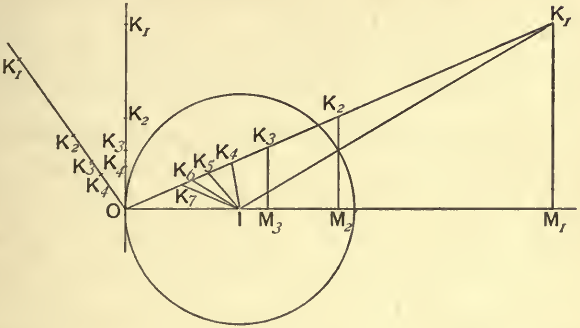
$$\left(1 + \frac{\nu^2}{1^2}\right)\left(1 + \frac{\nu^2}{2^2}\right)\left(1 + \frac{\nu^2}{3^2}\right) \, . \, . \, . \, ,$$

as the number of factors is increased without limit, is finite, being in fact, as will be shown hereafter (see Chap. x. § 7), $\frac{\sinh 2\nu}{\pi\nu}$; therefore the limit of the product $k_1 \, k_2 \ldots k_n$ is finite and the series is ultimately truly fluctuating, the radius of fluctuation being ultimately

$$\sqrt{\frac{\sinh 2\nu}{\pi\nu}} \, . \, \tfrac{1}{2} \sec \frac{u}{2} \text{ or } \sqrt{\frac{\eta^{2\nu} - \eta^{-2\nu}}{2\pi\nu}} \times \tfrac{1}{2} \sec \frac{u}{2},$$

which is finite except only in the case of $u = 4p + 2$ or $i^u = -1$.

The preceding investigation will be elucidated by the following geometrical interpretation :—



Take

$$\overline{OI} = 1,\ \overline{IM_1} = \mu,\ \overline{M_1\,K_1} = \nu i,$$

and therefore

$$\overline{IK_1} = \mu + \nu i \quad \text{and} \quad \overline{OK_1} = 1 + \mu + \nu i.$$

Then if $\overline{OK_2}$, $\overline{OK_3}$, $\overline{OK_4}\dots$ are respectively $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}\dots$ of $\overline{OK_1}$ or $1 + m$,

$$\overline{K_1I} = -m = 1 - \frac{1+m}{1} = k_1\, i^{\lambda_1},$$

$$\overline{K_2I} = 1 - \frac{1+m}{2} = k_2\, i^{\lambda_2},$$

$$\overline{K_3I} = 1 - \frac{1+m}{3} = k_3\, i^{\lambda_3},$$

and so on,

$$k_p{}^2 = IK_p{}^2 = OI^2 + OK_p{}^2 - 2OI\,.\,OM_p$$

$$= 1 + \frac{OK_1{}^2}{p^2} - 2\,\frac{OM_1}{p} = 1 - 2\,\frac{OM}{p}$$

nearly, when p is large,

and
$$n_m = .\,\overline{IK_1}\,.\,\overline{IK_2}\,.\;.\;.\,\overline{IK_n}.$$

Now however large $\overline{OM}$ or $\overline{\mu + 1}$ may be, only a finite number of K's fall without the unit circle, centre I, and for those within the circle IK_p, $IK_{p+1}\dots$ are all less than 1, tending to 1 as their limit, but the limit of their *product* is 0, since they differ from 1 by a quantity of the order $\frac{1}{p}$. Hence the remainder of the series after n terms tending always to the product of a fluctuating series and a zero radius of fluctuation is convergent, as long as $\mu + 1$ is positive.

If $\mu + 1 = 0$, M coincides with O, and K, K_1, $K_2 \ldots$ are all on the tangent to the unit circle at O and so are all without the circle, but IK_p tends to the limit 1 as p increases continually, differing from 1, when p is large, by a quantity of the order $\frac{1}{p^2}$, and consequently the limit of the product $\overline{IK_1} \,.\, \overline{IK_2} \ldots \overline{IK_n}$ is finite, so that the series is truly fluctuating, the limit of this product being the ultimate radius of fluctuation.

If $\mu + 1$ is negative, K, K_1, $K_2 \ldots$ are all without the circle on a secant, and IK_p is greater than 1 by a quantity of the order $\frac{1}{p}$, in which case the limit of the product $\overline{IK_1} \,.\, \overline{IK_2} \ldots \overline{IK_n}$ is infinite and the series is therefore divergent.

7. *Trigonometrical Series derived from the Binomial Theorem.*

If we replace i^u, &c., by De Moivre's form $\cos u + i \sin u$, &c., we find after rearrangement, and using the notation* 1_m, 2_m, $3_m \ldots$ for the coefficients of the series,

$$\rho^m (\cos mv_0 + i \sin mv_0) = 1 + 1_m r \cos u + 2_m r^2 \cos 2u + 3_m r^3 \cos 3u + \ldots$$
$$+ i \left\{1_m r \sin u + 2_m r^2 \sin 2u + 3_m r^3 \sin 3u + \ldots\right\},$$

whence

$$(1 + 2r \cos u + r^2)^{\frac{m}{2}} \cos mv_0 = 1 + 1_m r \cos u + 2_m r^2 \cos 2u + 3_m r^3 \cos 3u + \ldots$$

and

$$(1 + 2r \cos u + r^2)^{\frac{m}{2}} \sin mv_0 = 1_m r \sin u + 2_m r^2 \sin 2u + 3_m r^3 \sin 3u + \ldots,$$

true for all values of m, provided $r < 1$.

---

* This notation was suggested by De Morgan, to be read 1 *out of m*, 2 *out of m*, &c., being, when m is a positive integer, the number of combinations of 1, 2, ... things, out of a set of m things.

If

$$r=1,\ \rho=\left\{1+2r\cos u+r^2\right\}^{\frac{1}{2}}=\left\{2\,.\,\overline{1+\cos u}\right\}^{\frac{1}{2}}=2\cos\frac{u}{2},$$

$$\tan v_0=\frac{\sin u}{1+\cos u}=\tan\frac{u}{2},$$

and these series become (v_0 and therefore $\frac{u}{2}$ lying between -1 and $+1$),

$$2^m\left(\cos\frac{u}{2}\right)^m\cos\left(m\frac{u}{2}\right)=1+1_m\cos u+2_m\cos 2u+3_m\cos 3u+\,.\,.$$

$$2^m\left(\cos\frac{u}{2}\right)^m.\sin\left(m\frac{u}{2}\right)=1_m\sin u+2_m\sin 2u+3_m\sin 3u+\ldots$$

true, provided $m > -1$.

Ex. 1. Deduce from these, when m is a positive integer, the formulæ of Chap. IV. § 10.

Ex. 2. If m be the measure of an angle in right angles, prove from the above that

$$2^m\cos m=1-\frac{2m\,.\,\overline{2m-1}}{1\,.\,2}+\frac{2m\,.\,\overline{2m-1}\,.\,\overline{2m-2}\,.\,\overline{2m-3}}{1\,.\,2\,.\,3\,.\,4}-\&c.$$

$$2^m\sin m=2m-\frac{2m\,.\,\overline{2m-1}\,.\,\overline{2m-2}}{1\,.\,2\,.\,3}+\&c.,$$

and verify the result independently.

8. *Exponential Theorem.*

The Exponential Theorem asserts that for *all* values of z (scalar or complex), a^z is quantitatively equal to the infinite series

$$1+z\log_e a+\frac{z^2(\log_e a)^2}{1\,.\,2}+\ldots+\frac{z^n\log_e a)^n}{\underline{|n}}+\ldots$$

It may be deduced from the Binomial Theorem as follows :—

First Proof.

If x is a positive scalar < 1 (or a vector, whose tensor < 1), it is quantitatively true, as we have shown above, that

$$(1+x)^{\frac{z}{x}} = 1 + \frac{z}{x} . x + \frac{\frac{z}{x} . \left(\frac{z}{x} - 1\right)}{1 . 2} x^2 + \ldots$$

$$+ \frac{\frac{z}{x}\left(\frac{z}{x} - 1\right) \ldots \left(\frac{z}{x} - n + 1\right)}{1 . 2 \ldots n} x^n + \ldots$$

$$= 1 + z + \frac{z(z-x)}{1 . 2} + \ldots + \frac{z . (z-x) \ldots (z - \overline{n-1}x)}{1 . 2 \ldots n} + \ldots$$

whence, if u_n denote the n^{th} term of the series,

$$u_{n+1} = \frac{z - \overline{n-1}x}{n} . u_n,$$

and for any finite value of n,

$$\text{lt.}_{\,x=0} u_{n+1} = \frac{z}{n} . \text{lt.}_{\,x=0} u_n,$$

so that

$$\text{lt.}_{x=0} u_{n+1} = \frac{z}{n} . \frac{z}{n-1} \text{lt.}_{x=0} u_{n-1} = \frac{z}{n} . \frac{z}{n-1} . \frac{z}{n-2} \text{lt.}_{x=0} u_{n-2} = \ldots$$

$$= \frac{z}{n} . \frac{z}{n-1} \ldots \frac{z}{1} \text{lt.}_{x=0} u_1,$$

but $u_1 = 1$, therefore

$$\text{lt.}_{x=0} u_{n+1} = \frac{z^n}{1 . 2 . 3 \ldots n}.$$

Also since $u_{n+1} = -\left(x - \frac{z+x}{n}\right) u_n$, if n be taken so large that $\frac{z+x}{n}$ is very small compared with x, the sum of m terms after the

n^{th} term approaches without limit to $u_n(1-x+x^2\ldots(-x)^{m-1})$ or $u_n\dfrac{1-(-x)^m}{1+x}$, which ultimately $=\dfrac{z^n}{\underline{|n}}$, when $x=0$, and therefore is ultimately evanescent, when $x=0$ and n is infinite.

Hence $\text{lt.}_{x=0}(1+x)^{\frac{z}{x}}=1+z+\dfrac{z^2}{1\,.\,2}+\ldots+\dfrac{z^n}{\underline{|n}}+\ldots$

Putting $z=1$,

$$\text{lt.}_{x=0}(1+x)^{\frac{1}{x}}=1+1+\frac{1}{1\,.\,2}+\ldots+\frac{1}{\underline{|n}}+\ldots,$$

a series whose limit $(=2\cdot718281828\ldots)$ we have already denoted by e: whence, since

$$\left\{\text{lt.}_{x=0}(1+x)^{\frac{1}{x}}\right\}^z=\text{lt.}_{x=0}(1+x)^{\frac{z}{x}}.$$

$$e^z=1+z+\frac{z^2}{1\,.\,2}+\ldots+\frac{z^n}{\underline{|n}}+\ldots$$

and, since $a^z=(e^{\log_e a})^z=e^{z\log_e a}$,

$$a^z=1+z\log_e a+\frac{z^2(\log_e a)^2}{1\,.\,2}+\ldots+\frac{z^n(\log_e a)^n}{\underline{|n}}+\ldots$$

Second Proof.

The theorem may also be proved, assuming the Binomial Theorem for a positive integral index only, in the same manner as it was used in Chap. VI. to obtain the value of e.

We have seen that the $\text{limit}_{x=0}\dfrac{a^x-1}{x}=\log_e a$, so that if $a=e^z$, $\text{limit}_{x=0}\dfrac{e^{zx}-1}{x}=z$.

Let $\dfrac{\epsilon^{zx}-1}{x}=z$ and therefore $\epsilon^z=(1+zx)^{\frac{1}{x}}$, then e is the limit of ϵ and a that of ϵ^z, when x vanishes.

Let $x = \frac{1}{m}$, where m is a positive integer,

$$\text{then } \epsilon^z = \left(1 + \frac{z}{m}\right)^m = 1 + z + 1 \,.\left(1 - \frac{1}{m}\right)\frac{z^2}{\underline{|2}} + \ldots$$

$$+ 1 \,.\left(1 - \frac{1}{m}\right).\left(1 - \frac{2}{m}\right)\ldots\left(1 - \frac{r-1}{m}\right)\frac{z^r}{\underline{|r}} + \ldots$$

a finite series of $\overline{m+1}$ terms.

Now (see ch. v. § 7)

$$\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\ldots\left(1 - \frac{r-1}{m}\right) < 1 \text{ and } > 1 - \left(\frac{1}{m} + \frac{2}{m} + \ldots + \frac{r-1}{m}\right)$$

$$> 1 - \frac{r \,.\, \overline{r-1}}{2m},$$

it may therefore be taken to be equal to $1 - \theta_r \frac{r \,.\, \overline{r-1}}{2m}$, where θ_r is between 0 and 1.

$$\text{Hence } \epsilon^z = 1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \frac{z^4}{\underline{|4}} + \ldots + \frac{z^r}{\underline{|r}} + \ldots$$

$$- \frac{z^2}{2m}\left(1 + \theta_3 z + \theta_4 \frac{z^2}{1 \,.\, 2} + \ldots + \theta_r \frac{z^{r-2}}{\underline{|r-2}} + \ldots\right).$$

But the infinite series $1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \ldots$ is convergent for all values of z, and therefore $\theta_3, \theta_4 \ldots$ being all positive and less than 1, the series $1 + \theta_3 z + \theta_4 \frac{z^2}{1 \,.\, 2} + \ldots$ is also convergent, hence as m is increased without limit, the product $\frac{z^2}{2m}\left(1 + \theta_3 z + \theta_4 \frac{z^2}{1 \,.\, 2} + \ldots\right)$ ultimately vanishes, and as ϵ then becomes e, we have, for all values of z,

$$e^z = 1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \ldots + \frac{z^r}{\underline{|r}} + \ldots$$

whence, as before, $e = 1 + 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \ldots = 2{\cdot}718281828\ldots$

and $a^z = 1 + z\log_e a + \frac{z^2(\log_e a)^2}{1\,.\,2} + \ldots$

Third Proof.

The following proof, independent of the Binomial Theorem, though following the lines of Euler's Proof of that Theorem, is modified from Cauchy.

Consider the series $1 + z + \frac{z^2}{1\,.\,2} + \ldots + \frac{z^n}{\underline{|n}} + \ldots$, which we have shown (ch. viii. § 16 (2)) to be convergent for all values of z, and let it be denoted by $f(z)$.

Then also $f(y) = 1 + y + \frac{y^2}{1\,.\,2} + \ldots + \frac{y^n}{\underline{|n}} + \ldots$, and if the product $fy\,.\,fz$ be obtained by actual multiplication, it is easily seen that the term of the n^{th} degree in y, z, is

$$\frac{y^n}{\underline{|n}} + \frac{y^{n-1}}{\underline{|n-1}}\,.\,\frac{z}{1} + \frac{y^{n-2}}{\underline{|n-2}}\,.\,\frac{z}{\underline{|2}} + \ldots + \frac{y^{n-p}}{\underline{|n-p}}\,.\,\frac{z^p}{\underline{|p}} + \ldots + \frac{z^n}{\underline{|n}}$$

$$\text{or } \frac{1}{\underline{|n}}\left\{y^n + n\,.\,y^{n-1}z + \frac{n\,.\,n-1}{1\,.\,2}y^{n-2}z + \ldots + \frac{\underline{|n}}{\underline{|n-p}\,\underline{|p}}y^{n-p}z^p + \ldots + z^n\right\}$$

$$\text{or } \frac{1}{\underline{|n}}(y+z)^n,$$

so that $fy\,.\,fz = 1 + (y+z) + \frac{(y+z)^2}{\underline{|2}} + \ldots + \frac{(y+z)^n}{\underline{|n}} + \ldots$

$= f(y+z)$.

Let e be the value of fy, when $y = 1$, and therefore

$$e = 1 + 1 + \frac{1}{1\,.\,2} + \ldots + \frac{1}{\underline{|n}} + \ldots,$$

then $$f(z+1) = e\,.\,fz,$$

and $f(z+m) = e\,.\,f(z+m-1) = e^2 f(z+m-2) = \ldots = e^m fz$,

whence, putting $z=0$ and observing that $f(0)=1$,

$$fm = e^m,$$

or, if m is a positive integer,

$$e^m = 1 + m + \frac{m^2}{1\,.\,2} + \ldots + \frac{m^n}{\underline{|n}} + \ldots$$

Now this equation, being perfectly general in its form, must by the fundamental law of the permanence of equivalent forms be true for *any* value of m no less than for m restricted to be a positive integer, hence for any value of z whatsoever

$$e^z = 1 + z + \frac{z^2}{1\,.\,2} + \ldots + \frac{z^n}{\underline{|n}} + \ldots$$

It must be observed that in its most general sense e^z has an infinite number of distinct values for the same value of z, while the series above has but one value, but since the series is scalar, when z is so, that value of e^z must be taken in the equation above which becomes scalar with z.

Now (Chap. V. § 10) the general value of e^z is $e^x\,.\,e^{iy}\,.\,\eta^{-4\lambda y}\,.\,i^{4\lambda x}$, λ being any integer: and when $y=0$, so that $z=x$ (a scalar), it becomes $e^x i^{4\lambda x}$, which is scalar, only if $2\lambda x$ is integral; a condition which can only be satisfied in all cases, if $\lambda=0$. Hence the *fundamental* vector $e^x\,.\,e^{iy}$ or $e^x(\cos y + i \sin y)$ is that value of e^z, which is equal to the infinite series. This is the vector $\overline{OC}$ in the diagrams of Ch. V. §§ 10, 11, in the case where $OC = e^x$, $IOC = y$ radians, $OH = \eta^{-4y}$, $IOH = 4x$ right angles.

9. *Series (deduced from the Exponential Series) for sin θ, cos θ, sinh θ, cosh θ.*

Let $z = x + iy = r(\cos\theta + i\sin\theta)$, (using circular measure),

then $e^z = e^{r\cos\theta}\,.\,e^{ir\sin\theta} = e^{r\cos\theta}\{\cos(r\sin\theta) + i\sin(r\sin\theta)\}$

$$\text{and } e^z = 1 + r(\cos\theta + i\sin\theta) + \frac{r^2}{1\,.\,2}\,.\,(\cos 2\theta + i\sin 2\theta + \ldots)$$

$$= \left(1 + r\cos\theta + \frac{r^2}{1\,.\,2}\cos 2\theta + \ldots\right) + i\left(r\sin\theta + \frac{r^2}{1\,.\,2}\sin 2\theta + \ldots\right)$$

whence, equating projects and trajects separately in the two expressions for e^z,

$$e^{r\cos\theta}\,.\,\cos(r\sin\theta) = 1 + r\cos\theta + \frac{r^2}{1\,.\,2}\cos 2\theta + \ldots + \frac{r^n}{\underline{|n}}\cos n\theta + \ldots$$

and

$$e^{r\cos\theta}\sin(r\sin\theta) = r\sin\theta + \frac{r^2}{1\,.\,2}\sin 2\theta + \ldots + \frac{r^n}{\underline{|n}}\sin n\theta + \ldots,$$

in which, as in the exponential series, the series are convergent for *all* values of r.

From the above equations, putting $\theta = (4\lambda + 1)\frac{\pi}{2}$, so that $\cos\theta = 0$, $\sin\theta = 1$,

$$\cos n\theta = \cos \overline{4\lambda + 1}\,.\,n\frac{\pi}{2} = 0, \text{ if } n \text{ is odd, or } (-1)^{\frac{n}{2}}, \text{ if } n \text{ is even,}$$

$$\sin n\theta = \sin\left(\overline{4\lambda + 1}\,.\,n\frac{\pi}{2}\right) = 0, \text{ if } n \text{ is even, or} (-1)^{\frac{n-1}{2}}, \text{ if } n \text{ is odd,}$$

we have the following important series for the cosine and sine of an angle,

$$\cos r = 1 - \frac{r^2}{1\,.\,2} + \frac{r^4}{1\,.\,2\,.\,3\,.\,4} + \ldots + (-1)^n\frac{r^{2n}}{\underline{|2n}} - \ldots$$

$$\sin r = r - \frac{r^3}{\underline{|3}} + \frac{r^5}{\underline{|5}} - \ldots + (-1)^{n-1}\frac{r^{2n-1}}{\underline{|2n-1}} - \ldots$$

where r is the *circular measure* of the angle.

Again, since

$$e^z = 1 + z + \frac{z^2}{1\,.\,2} + \frac{z^3}{\underline{|3}} + \frac{z^4}{\underline{|4}} + \ldots$$

$$e^{-z}=1-z+\frac{z^2}{\underline{|2}}-\frac{z^3}{\underline{|3}}+\frac{z^4}{\underline{|4}}-\dots$$

and therefore

$$\cosh z=\frac{e^z+e^{-z}}{2}=1+\frac{z^2}{\underline{|2}}+\frac{z^4}{\underline{|4}}+\frac{z^6}{\underline{|6}}+\dots$$

$$\sinh z=\frac{e^z-e^{-z}}{2}=z+\frac{z^3}{\underline{|3}}+\frac{z^5}{\underline{|5}}+\frac{z^7}{\underline{|7}}+\dots$$

From these formulae for cosh z, sinh z, putting iz for z, we obtain, since cosh $iz=\cos z$ and sinh $iz=i\sin z$,

$$\cos z=1-\frac{z^2}{\underline{|2}}+\frac{z^4}{\underline{|4}}-\frac{z^6}{\underline{|6}}+\dots$$

$$\sin z=z-\frac{z^3}{\underline{|3}}+\frac{z^5}{\underline{|5}}-\frac{z^7}{\underline{|7}}+\dots$$

as we found otherwise above.

It is worth observing that, since sin z, cos z are periodic functions, the equivalent series are also periodic. It would be difficult to see *à priori* from the series themselves that they are periodic, but that they *may* be so is plain from the *alternating* character of the series; for if each term represent the length of a link of a chain, however much the successive links may increase in length at first, after reaching a certain link of maximum length they begin to diminish and ultimately become infinitesimally small, and it is plain that if such a chain is folded over into its most compact form, which is that represented by the series for sine and cosine, there may be different values of z, the quantity determining the lengths of the links, which will bring the second extremity of the chain ultimately to the same point at a finite distance from the first. The series for cosh z, sinh z are clearly non-periodic; in fact, they represent the chain stretched out to its full length.

10. *Logarithmic Series.*

We have seen (Ch. V. § 6) that $\mathrm{lt}_{x=0}\dfrac{a^x-1}{x}=\log_e a$: hence, if for a we write $1+z$, $\log_e(1+z)=\mathrm{lt}_{x=0}\dfrac{(1+z)^x-1}{x}$.

Now by the Binomial Theorem, if r, the modulus (or tensor) of z, < 1,

$$(1+z)^x-1=xz+\frac{x\,.\,\overline{x-1}}{1\,.\,2}z^2+\ldots+\frac{x\,.\,\overline{x-1}\ldots\overline{x-n+1}}{1\,.\,2\ldots n}z^n+\ldots$$

therefore

$$\frac{(1+z)^x-1}{x}=z-\frac{1\,.\,\overline{1-x}}{1\,.\,2}z^2+\frac{1\,.\,\overline{1-x}\,.\,\overline{2-x}}{1\,.\,2\,.\,3}z^3\ldots$$

$$+(-1)^{n-1}\frac{1\,.\,\overline{1-x}\ldots(\overline{n-1-x})}{1\,.\,2\,.\,3\ldots n}z^n+\ldots$$

$$=z-(1-x)\frac{z^2}{2}+\left(1-x\right)\left(1-\frac{x}{2}\right)\frac{z^3}{3}\ldots$$

$$+(-1)^{n-1}(1-x)\left(1-\frac{x}{2}\right)\ldots\left(1-\frac{x}{n-1}\right)\frac{z^n}{n}+\ldots$$

Now if in this series we put $x=0$, it reduces to

$$z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots;$$

but to show that, subject to the condition that the modulus of z is less than 1, this latter series is equal to $\log_e(1+z)$, the limiting value of $\dfrac{(1+z)^x-1}{x}$, when $x=0$, it must be proved that the former changes continuously, as x diminishes without limit. (See Ch. VIII. § 10.)

Now if P_n, Q_n denote the remainders of the two series after n terms,

$$P_n = (-1)^n (1-x)\left(1-\frac{x}{2}\right)\dots\left(1-\frac{x}{n}\right)\frac{z^{n+1}}{n+1}\left\{1-\left(1-\frac{x}{n+1}\right)\frac{n+1}{n+2}z\right.$$

$$\left.+\left(1-\frac{x}{n+1}\right)\left(1-\frac{x}{n+2}\right)\frac{n+1}{n+3}z^2-\dots\right\}$$

$$Q_n = (-1)^n . \frac{z^{n+1}}{n+1}\left\{1-\frac{n+1}{n+2}z+\frac{n+1}{n+3}.z^2-\dots\right\}$$

and the series factors of both P_n, Q_n ultimately approach without limit, as n is increased, to $1-z+z^2-z^3+\dots$ or, since Mod. $(z) < 1$, to the finite quantity $\frac{1}{1+z}$. Hence if we take n to denote such a number of terms that Mod. $Q_n = \epsilon$, a numerical quantity which we may make as small as we please,

$$\text{Mod. } P_n = \epsilon\,(1-x)\left(1-\frac{x}{2}\right)\dots\left(1-\frac{x}{n}\right)$$

very nearly, so that Mod. P_n and Mod. Q_n differ by a very small quantity for values of x near to 0. Hence the rates of convergency of the two series, measured by the number of terms necessary to be taken to get within a given distance ϵ from the limits of their sums, are very nearly equal, and therefore the second series is the limit of the first, when x vanishes, and we have

$$\log_e(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

quantitatively true, when Mod. $(z) < 1$. It is evident also that in the extreme case of $z=1$, the formula is true, the series being still convergent, so that

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots$$

but in the other extreme case of $z=-1$, it is not quantitatively true, the series factor of P_n, Q_n being in that case infinite, so that the reasoning above would not hold. The complete logarithm or logometer of a quantity is, as we have seen, a function having an infinite number of values, one only of which is scalar when z is scalar, and all are (vector or) complex, when z is so. Hence the series is equal to the *arithmetical* logarithm in the case of z scalar, and the fundamental or prime value of the logometer, which we may conveniently distinguish as the *logarithm*, and which is such that its traject lies between $-\pi$ and π, in the general case.

The series for $\log_e(1+z)$, being divergent for values of $z > 1$, can only be used directly for the calculation of logarithms (to the base e) of numbers not greater than 2, and this not conveniently on account of the slowness of its convergence. Other more convergent series however may be derived from it, as we will now concisely indicate.

Changing the sign of z, we have

$$\log_e(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \frac{z^5}{5} - \ldots$$

and since

$$\log_e \frac{1+z}{1-z} = \log_e(1+z) - \log_e(1-z)$$

$$\log_e \frac{1+z}{1-z} = 2\left\{z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\right\}.$$

If

$$\frac{1+z}{1-z} = \frac{m}{n},\ z = \frac{m-n}{m+n},$$

whence

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^3 + \ldots\right\}$$

a rapidly converging series, when m and n are considerable numbers differing by a much smaller number.

If $m = n+1$, we have

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3} \cdot \frac{1}{(2n+1)^3} + \frac{1}{5}\frac{1}{(2n+1)^5} + \ldots\right\}$$

a rapidly converging series for the difference of the Napierian logarithms of two consecutive numbers.

If $n=1$, we have

$$\log_e 2 = 2\left\{\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\frac{1}{7}\cdot\frac{1}{3^7}+\dots\right\}$$

which gives $\log_e 2 = \cdot 6931472\dots$ This the student should verify for himself, as also the following.

Taking $m=10$, $n=8$, $\frac{m-n}{m+n}=\frac{1}{9}$, and since $\log_e 8 = 3\log_e 2$,

$$\log_e 10 = 3\log_e 2 + 2\left\{\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{9^3}+\frac{1}{5}\cdot\frac{1}{9^5}+\dots\right\}$$

which gives $\log_e 10 = 2\cdot 3025851\dots$ and therefore

$$\log_{10} e = \frac{1}{\log_e 10} = \cdot 4342945\dots$$

This last number, the value of $\frac{1}{\log_e 10}$, is the factor by which Napierian logarithms must be multiplied in order to obtain the common logarithms to the base 10, and is termed the *modulus* of that system. To adapt the formulae above to the calculation of common logarithms, it is only necessary to prefix the factor μ $(=\cdot 4342945)$ to the series.

11. *Series (deduced from the Logarithmic Series) for* $\tan^{-1}x$, $\tanh^{-1}x$.

Let $$z = re^{\iota\theta} = r(\cos\theta + i\sin\theta)$$

and $$1+z = \rho e^{\iota\phi} = \rho(\cos\phi + i\sin\phi)$$

so that $$\rho\cos\phi = 1 + r\cos\theta,\ \ \rho\sin\phi = r\sin\theta,$$

and therefore $$\rho = \left\{1 + 2r\cos\theta + r^2\right\}^{\frac{1}{2}},$$

then $$\lambda_e(1+z) = \log_e\rho + i(\phi_0 + 2k\pi),$$

where ϕ_0 lies between $-\pi$ and π, and

$$\begin{aligned}\lambda_e(1+z) &= re^{i\theta} - \tfrac{1}{2}re^{2i\theta} + \tfrac{1}{3}re^{3i\theta} + \dots \\ &= r\cos\theta - \tfrac{1}{2}r^2\cos 2\theta + \tfrac{1}{3}r^3\cos 3\theta - \dots \\ &\qquad + i(r\sin\theta - \tfrac{1}{2}r^2\sin 2\theta + \tfrac{1}{3}r^3\sin 3\theta - \dots)\end{aligned}$$

if $r \not> 1$ and θ have any value except π or an odd multiple of π.

Hence, equating projects and trajects in the two expressions for $\lambda_e(1+z)$ and taking $k=0$ for the reasons given above,

$$\log_e\rho = \log_e\{1+2r\cos\theta+r^2\}^{\frac{1}{2}} = r\cos\theta - \tfrac{1}{2}r^2\cos 2\theta + \tfrac{1}{3}r^3\cos 3\theta - \dots$$

$$\phi_0 = \text{Tan}^{-1}\frac{r\sin\theta}{1+r\cos\theta} = r\sin\theta - \tfrac{1}{2}r^2\sin 2\theta + \tfrac{1}{3}r^3\sin 3\theta - \dots$$

where $r \not> 1$ and if $r=1$, θ has any value in the latter series, and any value except an odd multiple of π in the former.

If $\theta = \pm\frac{\pi}{2}$, $\rho\cos\phi_0 = 1$, and $\rho\sin\phi_0 = \pm r$, so that ϕ_0 lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$; also $\cos 2k\theta = \cos k\pi = (-1)^k$, $\cos(2k+1)\theta = 0$,

$$\sin 2k\theta = \sin k\pi = 0,\ \sin(2k+1)\theta = \pm(-1)^k,$$

and the above formulae reduce to

$$\log_e(1+r^2)^{\frac{1}{2}} = \frac{1}{2}r^2 - \frac{1}{4}r^4 + \frac{1}{6}r^6 - \dots$$

$$\text{Tan}^{-1} r = r - \frac{r^3}{3} + \frac{r^5}{5} - \frac{r^7}{7} + \dots$$

true for values of $r \not> 1$, whether positive or negative, so that $\text{Tan}^{-1} r$, denoting an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, must lie between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$.

The first affords only a verification of the general series above being obviously true from the series for $\log_e(1+z)$, but the other is an important result, known as Gregory's series for the circular measure of an angle in terms of its tangent.

The series may also be deduced thus through the series for $\tanh^{-1} x$. If

$$x = \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}, \quad e^{2z} = \frac{1+x}{1-x}$$

so that

$$z \text{ or } \tanh^{-1} x = \tfrac{1}{2} \log_e \frac{1+x}{1-x} + k\pi i,$$

where k is an integer, whence

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + k\pi i$$

or

$$\text{Tanh}^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

Mod. x not > 1.

Now

$$\tan^{-1} x = -i \tanh^{-1} ix$$

therefore

$$\tan^{-1} x = -i \,.\left\{ix - i\,\frac{x^3}{3} + i\,\frac{x^5}{5} - \ldots\right\} - k\pi i^2$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + k\pi$$

or

$$\text{Tan}^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots,$$

Mod. x not > 1.

12. *Calculation of the Value of* π.

Gregory's series furnishes the readiest methods of calculating the value of π.

If $x = 1$, since $\text{Tan}^{-1} 1 = \frac{\pi}{4}$, the series gives

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{2}{1\,.\,3}+\frac{2}{5\,.\,7}+\frac{2}{9.11}+\ldots$$

a series which converges too slowly to be convenient for calculation.

Since

$$\mathrm{Tan}^{-1}\frac{1}{2}+\mathrm{Tan}^{-1}\frac{1}{3}=\mathrm{Tan}^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\mathrm{Tan}^{-1}1=\frac{\pi}{4}$$

$$\frac{\pi}{4}=\left(\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}-\ldots\right)+\left(\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}-\ldots\right)$$

which is known as Euler's series, and is more rapidly convergent. (Eight terms of the first part and five of the second give the result correct to 6 figures. The student should verify this.)

A still more convergent series, known as Machin's, is obtained from the fact that $\frac{\pi}{4}$ differs little from 4 $\mathrm{Tan}^{-1}\frac{1}{5}$.

For

$$4\ \mathrm{Tan}^{-1}\frac{1}{5}=2\ \mathrm{Tan}^{-1}\frac{5}{12}=\mathrm{Tan}^{-1}\frac{120}{119},$$

and

$$\mathrm{Tan}^{-1}\frac{120}{119}-\mathrm{Tan}^{-1}1=\mathrm{Tan}^{-1}\frac{120-119}{120+119}=\mathrm{Tan}^{-1}\frac{1}{239},$$

therefore

$$\frac{\pi}{4}=4\ \mathrm{Tan}^{-1}\frac{1}{5}-\mathrm{Tan}^{-1}\frac{1}{239}.$$

The use of this formula is facilitated by observing that

$$\mathrm{Tan}^{-1}\frac{1}{239}=\mathrm{Tan}^{-1}\frac{1}{70}-\mathrm{Tan}^{-1}\frac{1}{99}$$

so that

$$\frac{\pi}{4}=4\ \mathrm{Tan}^{-1}\frac{1}{5}-\mathrm{Tan}^{-1}\frac{1}{70}+\mathrm{Tan}^{-1}\frac{1}{99}.$$

In 1853, Mr. Shanks published the value of π calculated to 607 decimal places, with which the independent calculation of Dr. Rutherford to 441 places agrees as far as it goes. The figures may be found in De Morgan's article in the *English Cyclopaedia*, "Quadrature of the Circle." To 20 places the result is

$$\pi = 3{\cdot}14159265358979323846\dots$$

13. *Series for* $\sin^{-1}x$, $\cos^{-1}x$, $\sinh^{-1}x$, $\cosh^{-1}x$.

These series can most readily be obtained by the use of the differential calculus, though they may be deduced from the series* in C. IV. § 9.

Since

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}} \text{ and } \frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{1+x^2}},$$

expanding by the Binomial Theorem, we have,

$$\frac{d}{dx}(\sin^{-1}x)=1+\frac{1}{2}x^2+\frac{1\,.\,3}{2\,.\,4}x^4+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6+\dots+\frac{1\,.\,3\dots(2n-1)}{2\,.\,4\dots 2n}x^{2n}$$
$$+\dots, \text{ if } x^2<1,$$

$$\frac{d}{dx}(\sinh^{-1}x)=1-\frac{1}{2}x^2+\frac{1\,.\,3}{2\,.\,4}x^4-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6+\dots$$
$$+(-1)^n.\frac{1\,.\,3\dots(2n-1)}{2\,.\,4\dots 2n}x^{2n}+\dots, \text{ if } x^2 \ngtr 1,$$

whence, integrating and observing that the value of $\sin^{-1}x$ between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ and the scalar value of $\sinh^{-1}x$ vanish with x, the circular measure of

$$\operatorname{Sin}^{-1}x=x+\frac{1}{2}\,.\,\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\,.\,\frac{x^5}{5}+\dots+\frac{1\,.\,3\dots(2n-1)}{2\,.\,4\dots 2n}\,.\,\frac{x^{2n+1}}{2n+1}+\dots$$

and

$$\operatorname{Sinh}^{-1}x=x-\frac{1}{2}\,.\,\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\,.\,\frac{x^5}{5}-\dots+(-1)^n\frac{1\,.\,3\dots(2n-1)}{2\,.\,4\dots 2n}\,.\,\frac{x^{2n+1}}{2n+1}+\dots$$

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* See Johnson's *Trigonometry*, Arts. 529 and 637.

Now these series are obviously convergent within the same limits as those whence they are derived, since the coefficients of the corresponding terms are less in the former than in the latter, so that from this it appears that the series for $\sinh^{-1} x$ is convergent both for $x^2 < 1$ and $x^2 = 1$, and that for $\sin^{-1} x$ for $x^2 < 1$. But it may be shown that this latter is also convergent for $x^2 = 1$, though the series whence it was derived is not so. For, when $x = 1$, the general term of the series is $\frac{1.3\ldots 2n-1}{2.4\ldots 2n} \cdot \frac{1}{2n+1}$, and from Wallis's Theorem (proved Ch. X. § 10) the limit of $\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\sqrt{2n+1}$, when $n = \infty$, is finite and $= \sqrt{\frac{2}{\pi}}$, therefore the general term approximates, as n increases, to

$$\sqrt{\frac{\pi}{2}} \cdot \frac{1}{(2n+1)^{\frac{3}{2}}},$$

and the remainder of the series after n terms approximates to a finite quantity $\times \left\{\frac{1}{(2n+1)^{\frac{3}{2}}} + \frac{1}{(2n+3)^{\frac{3}{2}}} + \frac{1}{(2n+5)^{\frac{3}{2}}} + \ldots\right\}$. But this series is the sum of certain of the terms of the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \ldots$ for the case of $m = \frac{3}{2}$, in which case, m being > 1, the series is convergent (Smith's Alg., Art. 270), so that the remainder of the series after n terms is convergent, and therefore the series itself convergent.

Hence the formulae above are both valid, if $x^2 < 1$ or $x^2 = 1$, and therefore for $\sin^{-1} x$ between the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ both inclusive, and for $\sinh^{-1} x$ between the limits $\sinh^{-1}(-1)$ or $\log_e(\sqrt{2}-1)$ and $\sinh^{-1} 1$ or $\log_e(\sqrt{2}+1)$, both inclusive.

Also, since $\mathrm{Cos}^{-1} x = \frac{\pi}{2} - \mathrm{Sin}^{-1} x$, and $\cos^{-1} x$ changes from 0 to π, while $\sin^{-1} x$ changes from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$,

$$\mathrm{Cos}^{-1} x = \frac{\pi}{2} - x - \frac{1}{2}\frac{x^3}{3} - \frac{1.3}{2.4} \cdot \frac{x^5}{5} - \ldots$$

where x may range from -1 to $+1$ inclusive.

Since $\mathrm{Cosh}^{-1} x = -i\,\mathrm{Cos}^{-1} x$, the above would give

$$\mathrm{Cosh}^{-1} x = i\left(-\frac{\pi}{2} + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\right)$$

true from $x = -1$ to $x + 1$ incl., corresponding to the fact that for values of x numerically less than 1, $\cosh^{-1} x$ is non-scalar. When $\cosh^{-1} x$ becomes scalar, the series becomes divergent.

To obtain a scalar series for $\cosh^{-1} x$, it may be expanded in a series of negative powers of x, thus:

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{d}{dx} \cdot \log_e (x + \sqrt{x^2 - 1}) = \frac{1}{\sqrt{x^2 - 1}} = \frac{1}{x}\left(1 - \frac{1}{x^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{x}\left\{1 + \frac{1}{2}\frac{1}{x^2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{x^4} + \dots\right\} \qquad \text{if } x^2 > 1$$

whence by integration, if $x^2 > 1$, and k some constant,

$$\cosh^{-1} x = k + \log_e x - \frac{1}{2} \cdot \frac{1}{2x^2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{4x^4} - \dots$$

$$- \frac{1 \cdot 3 \dots (2n-1)}{2 \cdot 4 \dots 2n} \cdot \frac{1}{2n}\frac{1}{x^{2n}} - \dots$$

If $x = \infty$, $k = \text{lt.}\{\cosh^{-1} x - \log_e x\} = \text{lt.}\{\log_e (x + \sqrt{x^2 - 1}) - \log_e x\}$

$$= \text{lt.} \log_e \left(1 + \sqrt{1 - \frac{1}{x^2}}\right)$$

$$= \log_e 2,$$

$$\text{therefore } \cosh^{-1} x = \log_e 2x - \frac{1}{2} \cdot \frac{1}{2x^2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{4x^4} \dots$$

$$- \frac{1 \cdot 3 \dots \overline{2n - 1}}{2 \cdot 4 \dots 2n} \cdot \frac{1}{2n}\frac{1}{x^n} - \&\text{c.}$$

as long as $\cosh^{-1} x$ is scalar.

This formula is valid not only for $x > 1$, but also for $x = 1$, since in that case $\cosh^{-1} x = 0$, and it reduces to

$$\log_e 2 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{6} + \&\text{c.}$$

and this series is convergent, as may be proved by considerations similar to those given above in that for $\sin^{-1} 1$.

14. *Summation by means of the preceding series.*

We have hitherto used the method explained in § 1 for the development of series from finite expressions: we shall now give some examples of its use for the summation of certain trigonometrical series.

(1) To sum the series $1 + \dfrac{\cos 2\theta}{\underline{|2}} + \dfrac{\cos 4\theta}{\underline{|4}} + \dfrac{\cos 6\theta}{\underline{|6}} + \dots$

Denote the sum of this series by U, then if V denote the sum of the conjugate series,

$$V = \frac{\sin 2\theta}{\underline{|2}} + \frac{\sin 4\theta}{\underline{|4}} + \frac{\sin 6\theta}{\underline{|6}} + \ldots$$

whence

$$U + iV = 1 + \frac{1}{\underline{|2}}(\cos 2\theta + i \sin 2\theta) + \frac{1}{\underline{|4}}(\cos 4\theta + i \sin 4\theta) + \ldots$$

$$= 1 + \frac{e^{2i\theta}}{\underline{|2}} + \frac{e^{4i\theta}}{\underline{|4}} + \ldots = \cosh\,(e^{i\theta}) \text{ by § 9.}$$

$$\text{Now } \cosh\,(e^{i\theta}) = \cosh\,(\cos\theta + i\sin\theta) = \frac{e^{\cos\theta} \,.\, e^{i\sin\theta} + e^{-\cos\theta} \,.\, e^{-i\sin\theta}}{2}$$

$$= \frac{e^{\cos\theta} + e^{-\cos\theta}}{2} \cos\,(\sin\theta) + i \frac{e^{\cos\theta} - e^{-\cos\theta}}{2} \sin\,(\sin\theta)$$

$$= \cosh\,(\cos\theta) \cos\,(\sin\theta) + i \sinh\,(\cos\theta) \,.\, \sin\,(\sin\theta).$$

$$\text{Hence } U = \cosh\,(\cos\theta) \cos \sin\theta \text{ or } \tfrac{1}{2}\,(e^{\cos\theta} + e^{-\cos\theta}) \,.\, \cos\,(\sin\theta)$$

$$V = \sinh\,(\cos\theta) \sin\,(\sin\theta) \text{ or } \tfrac{1}{2}\,(e^{\cos\theta} - e^{-\cos\theta}) \sin\,(\sin\theta),$$

so that we have summed the conjugate series as well as the given series.

(2) To sum the series $x \cos\theta + \frac{1}{3} x^3 \cos 3\theta + \frac{1}{5} x^5 \cos 5\theta + \ldots$, where $x \ngtr 1$.

$$\text{Taking } U = x\cos\theta + \tfrac{1}{3}x^3 \cos 3\theta + \tfrac{1}{5}x^5 \cos 5\theta + \ldots$$

$$\text{and } V = x\sin\theta + \tfrac{1}{3}x^3 \sin 3\theta + \tfrac{1}{5}x^5 \sin 5\theta + \ldots$$

$$U + iV = xe^{i\theta} + \tfrac{1}{3}x^3 e^{i\,.\,3\theta} + \tfrac{1}{5}x^5 \,.\, e^{i\,.\,5\theta} + \ldots$$

$$= \tanh^{-1}(xe^{i\theta}) \text{ by § 11.}$$

$$\text{Now } \tanh^{-1}(xe^{i\theta}) = \tfrac{1}{2} \log \frac{1 + xe^{i\theta}}{1 - xe^{i\theta}} = \tfrac{1}{2} \log \frac{1 + x\cos\theta + ix\sin\theta}{1 - x\cos\theta - ix\sin\theta}$$

or, making the denominator scalar, $= \frac{1}{2} \log \dfrac{(1 - x^2) + 2ix\sin\theta}{1 - 2x\cos\theta + x^2}$, so that $\tanh^{-1}(xe)^{i\theta} = \frac{1}{2}\log(\rho e^{i\phi}) = \frac{1}{2}\log\rho + i\,.\,\frac{\phi}{2},$

$$\text{if } \rho\cos\phi = \frac{1-x^2}{1-2x\cos\theta+x^2},\ \rho\sin\phi = \frac{2x\sin\theta}{1-2x\cos\theta+x^2}.$$

Hence

$$\rho^2 = \frac{(1-x^2)^2+4x^2\sin^2\theta}{(1-2x\cos\theta+x^2)^2} = \frac{(1+x^2)^2-4x^2\cos^2\theta}{(1-2x\cos\theta+x^2)^2} = \frac{1+2x\cos\theta+x^2}{1-2x\cos\theta+x^2}$$

$$\text{and } \tan\phi = \frac{2x\sin\theta}{1-x^2},$$

so that

$$U = \tfrac{1}{2}\log\rho = \tfrac{1}{4}\log\frac{1+2x\cos\theta+x^2}{1-2x\cos\theta+x^2} = \tfrac{1}{2}\tanh^{-1}\frac{2x\cos\theta}{1+x^2},$$

$$V = \tfrac{1}{2}\phi = \tfrac{1}{2}\tan^{-1}\frac{2x\sin\theta}{1-x^2}.$$

(3) To sum $\sin\theta + \sin 3\theta + \sin 5\theta + \dots$ to n terms.

Taking $V_n = \sin\theta + \sin 3\theta + \dots + \sin(2n-1)\theta,$

$$U_n = \cos\theta + \cos 3\theta + \dots + \cos(2n-1)\theta,$$

$$U_n + iV_n = e^{i\theta} + e^{3i\theta}\dots + e^{i(2n-1)\theta} = \frac{e^{i\theta}-e^{i(2n+1)\theta}}{1-e^{2i\theta}} = \frac{1-e^{2n\,.\,i\theta}}{e^{-i\theta}-e^{i\theta}}$$

$$= \frac{(1-\cos 2n\theta)-i\sin 2n\theta}{-2i\sin\theta} = \frac{\sin 2n\theta}{2\sin\theta} + i\,\frac{1-\cos 2n\theta}{2\sin\theta}.$$

Hence $U_n = \dfrac{\sin 2n\theta}{2\sin\theta},\quad V_n = \dfrac{1-\cos 2n\theta}{2\sin\theta} = \dfrac{\sin^2 n\theta}{\sin\theta}.$

Ex. 1. Deduce from the above that

$$1+3+5+\dots+(2n-1) = n^2.$$

Ex. 2. Illustrate the above by diagrams drawn for particular cases, e.g. $\theta = \frac{\pi}{4},\ \theta = \frac{\pi}{6}$, &c.

15. *Sum of selected terms of a known series.*

If the sum of the series $a_0 + a_1x + a_2x^2 + \ldots$ is a known function of x, the sum of every r^{th} term of the series beginning with any term, or the sum of $a_px^p + a_{p+r}x^{p+r} + a_{p+2r}x^{p+2r} + \ldots$, may be found by a general method dependent on the properties of the roots of unity (Ch. VII.), which will be understood, without a formal statement in general terms, by a few examples.

(1) To sum the series $1 + 3_nx^3 + 6_nx^6 + 9_nx^9 + \ldots$, where 1_n, 2_n, $3_n \ldots$ (read 1 out of n, 2 out of n, &c.) are used to denote the coefficients of x, x^2, $x^3 \ldots$ in the expansion of $(1+x)^n$.

The three cube roots of 1 are $i^{\frac{1}{3}}$, $i^{\frac{2}{3}}$, 1, and if we put $\omega = i^{\frac{1}{3}}$, then

$$1 + \omega + \omega^2 = 0,\ \ 1 + \omega^2 + \omega^4 = 0,\ \ 1 + \omega^3 + \omega^6 = 3, \ldots$$

and generally $1 + \omega^r + \omega^{2r} = 0$, unless r is a multiple of 3, in which case it $= 3$. We may apply these well-known results thus—

$$(1+x)^n = 1 + 1_nx + 2_nx^2 + 3_nx^3 + \ldots$$

$$(1+\omega x)^n = 1 + 1_n\omega x + 2_n\omega^2x^2 + 3_n\omega^3x^3 + \ldots$$

$$(1+\omega^2x)^n = 1 + 1_n\omega^2x + 2_n\omega^4x^2 + 3_n\omega^6x^3 + \ldots$$

Adding these, the coefficients of x, x^2, and any power of x not a multiple of 3 vanish, by the properties of ω stated above, so that

$$(1+x)^n + (1+\omega x)^n + (1+\omega^2x)^n = 3(1 + 3_nx^3 + 6_nx^6 + 9_nx^9 + \ldots)$$

$$\text{but } \omega = i^{\frac{1}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$\text{so that } 1 + \omega x = 1 - \frac{x}{2} + i \,.\, \frac{x\sqrt{3}}{2}$$

$$= (1 - x + x^2)^{\frac{1}{2}}(\cos\phi + i\sin\phi), \text{ where } \tan\phi = \frac{x\sqrt{3}}{2-x}.$$

So also, since $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$,

$$1 + \omega^2x = (1 - x + x^2)^{\frac{1}{2}}(\cos\phi - i\sin\phi),$$

Hence $(1+\omega x)^n + (1+\omega^2 x)^n = (1 - x + x^2)^{\frac{n}{2}} \,.\, 2\cos n\phi$,

and therefore

$$1 + 3_n x^3 + 6_n x^6 + \ldots$$

$$= \frac{1}{3}(1+x)^n + \frac{2}{3}(1 - x + x^2)^{\frac{n}{2}} \cos\left\{n \,.\, \tan^{-1}\frac{x\sqrt{3}}{2-x}\right\}.$$

If $x = 1$, $1 + 3_n + 6_n + 9_n + \ldots = \dfrac{2^n}{3} + \dfrac{2}{3}\cos\left(n\dfrac{\pi}{3}\right)$,

a result which it would be well for the student to verify by particular values of n.

(2) To sum $x + \dfrac{x^5}{\underline{|5}} + \dfrac{x^9}{\underline{|9}} + \ldots$

Here the sum of every fourth term of the series for ϵ^x, beginning with the 2nd, is required.

The four 4th roots of 1 are i, 1, $-i$, 1,

$$\text{also } x^3 \,.\, \epsilon^x = x^3 + \frac{x^4}{\underline{|1}} + \frac{x^5}{\underline{|2}} + \frac{x^6}{\underline{|3}} + \frac{x^7}{\underline{|7}} + \frac{x^8}{\underline{|8}} + \ldots$$

and writing successively ix, $-x$, $-ix$ for x, we have, since $i^2 = -1$, $i^3 = -i$, &c.

$$-ix^3\epsilon^{ix} = -ix^3 + \frac{x^4}{\underline{|1}} + i\frac{x^5}{\underline{|2}} - \frac{x^6}{\underline{|3}} - i\frac{x^7}{\underline{|4}} + \frac{x^8}{\underline{|5}} + \ldots$$

$$-x^3\epsilon^{-x} = -x^3 + \frac{x^4}{\underline{|1}} - \frac{x^5}{\underline{|2}} + \frac{x^6}{\underline{|3}} - \frac{x^7}{\underline{|4}} + \frac{x^8}{\underline{|5}} + \ldots$$

$$ix^3\epsilon^{-ix} = ix^3 + \frac{x^4}{\underline{|1}} - i\frac{x^5}{\underline{|2}} - \frac{x^6}{\underline{|3}} + i\frac{x^7}{\underline{|4}} + \frac{x^8}{\underline{|5}} + \ldots$$

whence by addition

$$x^3\left\{\epsilon^x - \epsilon^{-x} - i\left(\epsilon^{ix} - \epsilon^{-ix}\right)\right\} = 4x^3\left\{x + \frac{x^5}{\underline{|5}} + \frac{x^9}{\underline{|9}} + \ldots\right\}$$

$$\text{or } 2\{\sinh x - i^2 \sin x\} = 4\left\{x + \frac{x^5}{\underline{|5}} + \frac{x^9}{\underline{|9}} + \ldots\right\}$$

$$\text{or } \frac{1}{2}(\sinh x + \sin x) = x + \frac{x^5}{\underline{|5}} + \frac{x^9}{\underline{|9}} + \ldots$$

a result easily verified from the expansions in § 9.

(3) To sum $x + \frac{x^7}{7} + \frac{x^{13}}{13} + \frac{x^{19}}{19} + \ldots$

we have

$$x^{-1} \,.\, \log(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7} - \ldots$$

Now, if $\omega = i^{\frac{4}{6}} = i^{\frac{2}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, the sixth-roots of 1 are

$$1,\ \omega,\ \omega^2,\ -1,\ -\omega,\ -\omega^2,$$

and substituting in the above formula ωx, $\omega^2 x$, $-x$, $-\omega x$, $-\omega^2 x$, and adding,

$$\frac{1}{x}\{\log(1+x) + \omega^{-1}\log(1+\omega x) + \omega^{-2}\log(1+\omega^2 x) - \log(1-x)$$
$$- \omega^{-1}\log(1-\omega x) - \omega^{-2}\log(1-\omega^2 x)\}$$
$$= 6\left\{1 + \frac{x^6}{7} + \frac{x^{12}}{13} + \ldots\right\}$$

The left-hand side within the bracket reduces to

$$\log\frac{1+x}{1-x} + \omega^{-1}\log\frac{1+\omega x}{1-\omega x} + \omega^{-2}\log\frac{1+\omega^2 x}{1-\omega^2 x},$$

$$\text{or } \left(\text{since } \omega^{-1} = \frac{1}{2} - i\frac{\sqrt{3}}{2},\ \omega^{-2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), \text{ to}$$

$$\log\frac{1+x}{1-x} + \frac{1}{2}\log\frac{(1+\omega x)(1-\omega^2 x)}{(1-\omega x)(1+\omega^2 x)} + i\frac{\sqrt{3}}{2}\log\frac{(1-\omega x)(1-\omega^2 x)}{(1+\omega x)(1+\omega^2 x)},$$

but $\qquad (1+\omega x)(1-\omega^2 x) = 1 + \overline{\omega - \omega^2}x - \omega^3 x^2 = 1 + x + x^2$

and $$(1-\omega x)(1+\omega^2 x)=1-x+x^2$$

and $$(1-\omega x)(1-\omega^2 x)=1-(\omega+\omega^2)x+\omega^3 x^2=1-x^2-ix\sqrt{3}$$

and $$(1+\omega x)(1+\omega^2 x)=1-x^2+ix\sqrt{3},$$

therefore the expression farther reduces to

$$\log\left(\frac{1+x}{1-x}\right)+\frac{1}{2}\log\frac{1+x+x^2}{1-x+x^2}+i\,\frac{\sqrt{3}}{2}\log\frac{1-x^2-ix\sqrt{3}}{1-x^2+ix\sqrt{3}}$$

or

$$\tfrac{1}{2}\log\frac{(1+x)^2(1+x+x^2)}{(1-x)^2(1-x+x^2)}+\sqrt{3}\,\mathrm{Tan}^{-1}\frac{x\sqrt{3}}{1-x^2},$$

so that

$$x+\frac{x^7}{7}+\frac{x^{13}}{13}+\frac{x^{19}}{19}+\dots$$
$$=\frac{1}{12}\log\frac{(1+x)^2(1+x+x^2)}{(1-x)^2(1-x+x^2)}+\frac{\sqrt{3}}{6}\,.\,\tan^{-1}\frac{x\sqrt{3}}{1-x^2},$$

where $x<1$. If $x=1$ or >1, the series is divergent and cannot be summed.

Ex. Shew that

$$1-\frac{1}{7}+\frac{1}{13}-\frac{1}{19}+\dots=\frac{\sqrt{3}}{12}\,.\log(7+4\sqrt{3}).$$

16. *Summation of Trigonometrical Series.*

Certain Trigonometrical Series can be summed by the general method of differences, the application of which to Algebraical Series the student will have seen in ordinary Algebra (see C. Smith's *Algebra*, Art. 313). If u_n is the n^{th} term of a series,

$$u_1+u_2+\dots+u_n,$$

and u_n can be expressed as the difference of two consecutive values of another function (v) for the values $n-1$ and n of the variable, so that these values being denoted by v_{n-1}, v_n respectively, $u_n=v_n-v_{n-1}$, then

$$u_1 = v_1 - v_0,\ u_2 = v_2 - v_1,\ u_3 = v_3 - v_2 \ldots u_n = v_n - v_{n-1},$$

and therefore

$$u_1 + u_2 + \ldots + u_n = v_n - v_0,$$

so that a finite expression is obtained for the sum of n terms. The full development of this subject is to be found in treatises on the "Calculus of Finite Differences." We will shew by a few examples how the method may be applied to obtain series from certain finite expressions, and inversely to sum certain series.

1. Take the formula

$$\sin\phi - \sin\theta = 2\cos\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}.$$

Then if $\phi - \theta = 2a$

$$\sin(\theta + 2a) - \sin\theta = 2\cos(\theta + a)\sin a,$$

and so

$$\sin(\theta + 4a) - \sin(\theta + 2a) = 2\cos(\theta + 3a)\sin a,$$

. . . .

$$\sin(\theta + 2na) - \sin(\theta + \overline{2n-1a}) = 2\cos(\theta + \overline{2n-1a})\sin a,$$

whence

$$\sin(\theta + 2na) - \sin\theta = 2\sin a\left\{\cos\overline{\theta + a} + \cos\overline{\theta + 3a} + \ldots + \cos(\theta + \overline{2n-1})\,a\right\},$$

or

$$\frac{\sin na}{\sin a}.\cos(\theta + na) = \cos(\theta + a) + \cos(\theta + 3a) + \ldots + \cos(\theta + \overline{2n-1}a).$$

Hence, if $\theta = 0$, $\cos a + \cos 3a + \ldots + \cos\overline{2n-1}a = \dfrac{\sin(2na)}{2\sin a}$,

and if $\theta = \dfrac{\pi}{2}$, $\sin a + \sin 3a \ldots + \sin\overline{2n-1}a = \dfrac{\sin^2(na)}{\sin a}$.

Again, if $na = \pi$,

$$\cos\left(\theta + \frac{\pi}{n}\right) + \cos\left(\theta + \frac{3\pi}{n}\right) + \ldots + \cos\left(\theta + \overline{2n-1}\,\frac{\pi}{n}\right) = 0,$$

$$\sin\left(\theta+\frac{\pi}{n}\right)+\sin\left(\theta+\frac{3\pi}{n}\right)+\ldots+\sin\left(\theta+\overline{2n-1}\,\frac{\pi}{n}\right)=0.$$

Ex. Verify these last results by a diagram, and the previous results by taking $\alpha=0$.

2. From the formula

$$\tan\phi-\tan\theta=\frac{\sin(\phi-\theta)}{\cos\phi\cos\theta}$$

if $\theta-\phi=\alpha$,

$$\tan(\theta+\alpha)-\tan\theta=\sin\alpha\sec\theta\,.\,\sec\theta+\alpha,$$

so also

$$\tan(\theta+2\alpha)-\tan(\theta+\alpha)=\sin\alpha\sec(\theta+\alpha)\sec(\theta+2\alpha)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tan(\theta+n\alpha)-\tan(\theta+\overline{n-1}\alpha)=\sin\alpha\sec(\theta+\overline{n-1}\alpha)\,.\,\sec(\theta+n\alpha),$$

whence

$$\tan(\theta+n\alpha)-\tan\theta=\sin\alpha\left\{\sec\theta\,.\,\sec\overline{\theta+\alpha}\right.$$

$$\left.+\sec\overline{\theta+\alpha}\,.\,\sec\overline{\theta+2\alpha}+\ldots+\sec(\theta+\overline{n-1}\alpha)\sec(\theta+n\alpha)\right\},$$

or

$$\frac{\sin n\alpha}{\sin\alpha}\sec\theta\,.\,\sec(\theta+n\alpha)=\sec\theta\,.\,\sec\overline{\theta+\alpha}$$

$$+\sec\overline{\theta+\alpha}\sec\overline{\theta+2\alpha}+\ldots+\sec(\theta+\overline{n-1}\alpha)\,.\,\sec(\theta+n\alpha).$$

3. To sum the series $\sin\theta+\sin(\theta+\alpha)+\sin(\theta+2\alpha)+\ldots$ to n terms. If S_n denote the sum,

$$2S_n\sin\frac{\alpha}{2}=2\sin\theta\sin\frac{\alpha}{2}+2\sin(\theta+\alpha)\sin\frac{\alpha}{2}+\ldots$$

$$+2\sin(\theta+\overline{n-1}\alpha)\sin\frac{\alpha}{2}$$

$$=\left\{\cos\overline{\theta-\frac{\alpha}{2}}-\cos\overline{\theta+\frac{\alpha}{2}}\right\}+\left\{\cos\overline{\theta+\frac{\alpha}{2}}-\cos\overline{\theta+\frac{3\alpha}{2}}\right\}$$

$$+\ldots+\left\{\cos\left(\theta+\overline{2n-3}\,\frac{\alpha}{2}\right)-\cos\left(\theta+\overline{2n-1}\,\frac{\alpha}{2}\right)\right\}$$

$$= \cos\left(\theta - \frac{a}{2}\right) - \cos\left(\theta + \overline{2n-1}\,\frac{a}{2}\right)$$

$$= 2\sin\frac{na}{2} \cdot \sin\left\{\theta + \overline{n-1}\,\frac{a}{2}\right\},$$

or

$$\sin\theta + \sin(\theta + a) + \ldots + \sin(\theta + \overline{n-1}a) = \frac{\sin\frac{na}{2}}{\sin\frac{a}{2}} \cdot \sin\left(\theta + \overline{n-1}\,\frac{a}{2}\right)$$

Hence, putting $\frac{\pi}{2} + \theta$ for θ, we obtain the conjugate series,

$$\cos\theta + \cos(\theta + a) + \ldots + \cos(\theta + \overline{n-1}a) = \frac{\sin\frac{na}{2}}{\sin\frac{a}{2}} \cdot \cos\left(\theta + \overline{n-1}\,\frac{a}{2}\right).$$

These results could also have been easily obtained by the method of § 13, and the results of Ex. 3 in § 13 are easily deducible from those above.

Ex. Let $\overline{OA_1}$ be a unit vector inclined to the prime vector at the angle θ: describe the circle with the chord OA_1 subtending the angle a at its centre: take successive chords

$$A_1 A_2,\ A_2 A_3 \ldots A_{n-1} A_n,$$

each equal to OA_1: then from the value of $\overline{OA_n}$ obtained from the figure the above results may be at once deduced.

4. To sum

$$\operatorname{cosec}\theta + \operatorname{cosec}\frac{\theta}{2} + \ldots + \operatorname{cosec}\frac{\theta}{2^n}.$$

Since
$$\operatorname{cosec}\theta + \cot\theta = \frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2},$$

$$\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta, \text{ and } \operatorname{cosec}\frac{\theta}{2^n} = \cot\frac{\theta}{2^n} - \cot\frac{\theta}{2^{n-1}}$$

whence by addition

$$\operatorname{cosec}\theta+\operatorname{cosec}\frac{\theta}{2}+\ldots+\operatorname{cosec}\frac{\theta}{2^n}=\cot\frac{\theta}{2^n}-\cot\theta.$$

5. To sum

$$\tanh x+\frac{1}{2}\tanh\frac{x}{2}+\frac{1}{2^2}\tanh\frac{x}{2^2}+\ldots \text{ to } n \text{ terms.}$$

Since

$$\tanh x+\coth x=\frac{\sinh^2 x+\cosh^2 x}{\sinh x\,.\,\cosh x}=\frac{\cosh 2x}{\frac{1}{2}\sinh 2x}=2\coth 2x$$

$$\tanh x=2\coth 2x-\coth x,\ \frac{1}{2}\tanh\frac{x}{2}=\coth x-\frac{1}{2}\coth\frac{x}{2}.\ \ldots,$$

$$\frac{1}{2^{n-1}}\tanh\frac{x}{2^{n-1}}=\frac{1}{2^{n-2}}\coth\frac{x}{2^{n-2}}-\frac{1}{2^{n-1}}\coth\frac{x}{2^{n-1}},$$

therefore

$$\tanh x+\frac{1}{2}\tanh\frac{x}{2}+\frac{1}{2^2}\tanh\frac{x}{2^2}+\ldots$$

$$+\frac{1}{2^{n-1}}\tanh\frac{x}{2^{n-1}}=2\coth 2x-\frac{1}{2^{n-1}}\coth\frac{x}{2^{n-1}}.$$

Since

$$\frac{1}{2^{n-1}}\coth\frac{x}{2^{n-1}}=\frac{1}{x}\cdot\frac{\theta}{\tanh\theta} \text{ if } \theta=\frac{x}{2^{n-1}},$$

and when

$$n=\infty,\ \theta=0 \text{ and lt.}_{\theta=0}\frac{\theta}{\tanh\theta}=1,$$

$$\tanh x+\frac{1}{2}\tanh\frac{x}{2}+\frac{1}{2^2}\tanh\frac{x}{2^2}+\ldots \text{ ad inf.}=2\coth 2x-\frac{1}{x}.$$

Ex. 1. Verify this result by taking $x=0$.

Ex. 2. What does this become, if ix is put for x?

17. *Bernoulli's Numbers.*

The coefficients of the expansion of $\frac{x}{e^x-1}$ form a series of numbers, known as Bernoulli's numbers (having been first used by James Bernoulli in his work *Ars Conjectandi*, posthumously published at Bâle, 1713), which are of great utility in the expansion of certain functions and the summation of certain series.

It should first be proved that it is possible to expand $\frac{x}{e^x-1}$ in a convergent series of positive powers of x within certain limits of the value of x.

Since

$$\frac{x}{e^x-1}=\frac{x}{x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots}=\frac{1}{1+\left(\frac{x}{\underline{|2}}+\frac{x^2}{\underline{|3}}+\frac{x^3}{\underline{|4}}+\ldots\right)}$$

and the infinite series in the denominator is convergent for all values of x; if limits of x can be found for which

$$\frac{x}{2}+\frac{x^2}{\underline{|3}}+\frac{x^3}{\underline{|4}}+\ldots<1$$

numerically, $\frac{x}{e^x-1}$ can be expanded in the form

$$1-\left(\frac{x}{\underline{|2}}+\frac{x^2}{\underline{|3}}+\ldots\right)+\left(\frac{x}{\underline{|2}}+\frac{x^2}{\underline{|3}}+\ldots\right)^2-\left(\frac{x}{\underline{|2}}+\frac{x^2}{\underline{|3}}+\ldots\right)^3+\ldots$$

which is then *absolutely* convergent, since it remains so, when all its terms are taken as positive.

Now $\frac{x}{2}+\frac{x^2}{\underline{|3}}+\frac{x^3}{\underline{|4}}+\ldots$ or $\frac{e^x-1-x}{x}<1$, if $e^x<1+2x$: but taking $x=1$, we know that $e<3$, and taking $x=2$, $e^2>5$, so that for some value between 1 and 2, which will be found by trial to be very nearly 1.25, $e^x = 1 + 2x$: hence, if $x < 1.25$, the above series is certainly convergent, and then if the terms were expanded and combined so as to form a series arranged according to powers of x, a convergent series of the required form would be obtained.

This proof only shews that a convergent series can certainly be found, if $x < 1.25$, but it may be that the limit of the value of x for convergency is greater than this, and we shall later on prove it to be so.*

18. *Expansion of* $\frac{x}{e^x - 1}$.

Let then

$$\frac{x}{e^x - 1} = B_0 + B_1 x + B_2 \frac{x^2}{\underline{|2}} + B_3 \frac{x^3}{\underline{|3}} + B_4 \frac{x^4}{\underline{|4}} + \ldots,$$

where B_0, B_1, $B_2 \ldots$ are numbers to be determined.

Changing the sign of x, we have $\frac{-x}{e^{-x} - 1} = \frac{xe^x}{e^x - 1}$, so that

$$\frac{xe^x}{e^x - 1} = B_0 - B_1 x + B_2 \frac{x^2}{\underline{|2}} - B_3 \frac{x^3}{\underline{|3}} + \ldots$$

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* The beginner might be tempted to proceed thus. Since $e^{-x} < 1$,

$$\frac{x}{e^x - 1} = \frac{xe^{-x}}{1 - e^{-x}} = x(e^{-x} + e^{-2x} + e^{-3x} + \ldots),$$

a convergent series, and the series for e^{-x}, $e^{-2x} \ldots$ are all convergent, and therefore that for $e^{-x} + e^{-2x} + e^{-3x} + \ldots$ is so. But on forming this series we find it takes the shape

$$(1 + 1 + 1 + \ldots) - (1 + 2 + 3 + \ldots) x + (1^2 + 2^2 + 3^2 + \ldots) \frac{x^2}{1 \,.\, 2} - \&c.,$$

a series of positive and negative infinite terms.

This example shews the need of great caution in dealing with doubly infinite series. On this point the student is referred to Chrystal's *Algebra*, chap. xxvi., §§ 32—34, and chap. xxviii., § 6.

and by subtraction,

$$\frac{xe^x - x}{e^x - 1} \text{ or } x = -2B_1 x - 2B_3 \frac{x^3}{\underline{|3}} - 2B_5 \frac{x^5}{\underline{|5}} - \dots$$

whence $B_1 = -\frac{1}{2}$, and $B_3, B_5 \dots B_{2r-1}$, are all zero. We may therefore write, B_0 being obviously 1,

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + B_2 \frac{x^2}{\underline{|2}} + B_4 \frac{x^4}{\underline{|4}} + B_6 \frac{x^6}{\underline{|6}} \dots + B_{2r} \frac{x^{2r}}{\underline{|2r}} + \dots$$

or, more conveniently, anticipating the result that the coefficients are alternately positive and negative,

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + B_2 \frac{x^2}{\underline{|2}} - B_4 \frac{x^4}{\underline{|4}} + \dots + (-1)^{r-1} B_{2r} \frac{x^{2r}}{\underline{|2r}} + \dots$$

Also

$$\frac{e^x - 1}{x} = 1 + \frac{x}{\underline{|2}} + \frac{x^2}{\underline{|3}} + \frac{x^3}{\underline{|4}} + \frac{x^4}{\underline{|5}} + \dots + \frac{x^{2r}}{\underline{|2r+1}} + \dots$$

Hence the coefficients of all the powers of x in the product of these two series must vanish, so that, selecting the coefficient of x^{2r},

$$\frac{1}{\underline{|2r+1}} - \frac{1}{2}\frac{1}{\underline{|2r}} + \frac{B_2}{\underline{|2r-1}\,\underline{|2}} - \frac{B_4}{\underline{|2r-3}\,\underline{|4}} + \dots$$

$$- (-1)^{r-1} \frac{B_{2r-2}}{\underline{|3}\,\underline{|2r-2}} + (-1)^{r-1} \frac{B_{2r}}{\underline{|2r}} = 0,$$

and therefore

$$B_{2r} = B_{2r-2} \frac{2r \,.\, \overline{2r-1}}{\underline{|3}} - B_{2r-4} \frac{\overline{2r} \,.\, \overline{2r-1} \,.\, \overline{2r-2} \,.\, \overline{2r-3}}{\underline{|5}} + \dots$$

$$+ (-1)^r B_2 \frac{2r}{\underline{|2}} - \frac{(-1)^r}{2} + (-1)^r \frac{1}{2r+1},$$

a formula whence, by putting r successively equal to 1, 2, 3, 4..., the values of $B_2, B_4, B_6 \dots$ may be found.

The result as far as B_{14} (which the student should verify) is

$$B_2=\frac{1}{6},\ B_4=\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=\frac{1}{30},\ B_{10}=\frac{5}{66},\ B_{12}=\frac{691}{2730},\ B_{14}=\frac{7}{6}.$$

Ex. Shew also that

$$B_{2r}=B_{2r-2}\frac{2r\,.\,\overline{2r-1}}{3\,.\,4}-B_{2r-4}\frac{2r\,.\,\overline{2r-1}\,.\,\overline{2r-2}\,.\,\overline{2r-3}}{3\,.\,4\,.\,5\,.\,6}+\dots$$

$$+(-1)^r B_2-\frac{(-1)^r}{2r+1}+\frac{(-1)^r}{(r+1)(2r+1)}.$$

19. *Series for coth x, cot x, tanh x, tan x, cosech x, cosec x.*

We have proved that

$$\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+B_2\frac{x}{\underline{|2}}-\frac{B_4}{\underline{|4}}\,.\,x^3+\frac{B_6}{\underline{|6}}x^5-\dots$$

Now $$\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{e^{2x}+1}{e^{2x}-1}=1+\frac{2}{e^{2x}-1},$$

therefore

$$\coth x=1+2\left\{\frac{1}{2x}-\frac{1}{2}+\frac{B_2}{\underline{|2}}\,.\,2x-\frac{B_4}{\underline{|4}}\,.\,2^3x^3+\frac{B_6}{\underline{|6}}\,.\,2^5x^5+\dots\right\}$$

$$=\frac{1}{x}+\frac{B_2}{\underline{|2}}\,.\,2^2x-\frac{B_4}{\underline{|4}}\,.\,2^4x^3+\frac{B_6}{\underline{|6}}\,.\,2^6x^5-\dots$$

Hence, putting ix in the place of x, since $i\coth ix=\cot x$,

$$\cot x=\frac{1}{x}-\frac{B_2}{\underline{|2}}2^2x-\frac{B_4}{\underline{|4}}2^4x^3-\frac{B_6}{\underline{|6}}2^6x^5-\dots$$

Also $$\coth x+\tanh x=\frac{\cosh^2 x+\sinh^2 x}{\cosh x\,.\,\sinh x}=\frac{\cosh 2x}{\frac{1}{2}\sinh 2x}=2\coth 2x,$$

therefore $$\tanh x=2\coth 2x-\coth x,$$

and in like manner $$\tan x=\cot x-2\cot 2x,$$

whence

$$\tanh x = 2\left\{\frac{1}{2x} + \frac{B_2}{\underline{|2}}2^3x - \frac{B_4}{\underline{|4}}\,.\,2^7x^3 + \frac{B_6}{\underline{|6}}\,.\,2^{11}x^5 - \ldots\right\}$$

$$- \left\{\frac{1}{x} + \frac{B_2}{\underline{|2}}\,2^2x - \frac{B_4}{\underline{|4}}2^4x^3 + \ldots\right\},$$

$$\text{or } \tanh x = \frac{B_2}{\underline{|2}}\,2^2\,(2^2 - 1)\,x - \frac{B_4}{\underline{|4}}\,.\,2^4\,(2^4 - 1)\,x^3 + \frac{B_6}{\underline{|6}}\,.\,2^6\,(2^6 - 1)x^5 - \ldots$$

and

$$\tan x = \frac{B_2}{\underline{|2}}\,.\;2^2\,(2^2 - 1)\,x + \frac{B_4}{\underline{|4}}\,2^4\,(2^4 - 1)\,x^3$$

$$+ \frac{B_6}{\underline{|6}}\,.\,2^6\,.\,(2^6 - 1)x^5 + \ldots$$

Substituting the numerical values of B_2, $B_4 \ldots$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$

Again

$$\coth x - \tanh x = \frac{\cosh^2 x - \sinh^2 x}{\sinh x \cosh x} = 2 \operatorname{cosech} 2x,$$

therefore

$$\operatorname{cosech} x = \tfrac{1}{2}\left(\coth\frac{x}{2} - \tanh\frac{x}{2}\right)$$

$$= \frac{1}{x} + \frac{B_2}{\underline{|2}}x - \frac{B_4}{\underline{|4}}x^3 + \frac{B_6}{\underline{|6}}x^5 - \ldots$$

$$- \left\{\frac{B_2}{\underline{|2}}(2^2 - 1)\,x - \frac{B_4}{\underline{|4}}(2^4 - 1)\,x^3 + \frac{B_6}{\underline{|6}}(2^6 - 1)\,x^5 - \ldots\right\}$$

$$= \frac{1}{x} - \frac{B_2}{\underline{|2}}\,.\,2\,(2 - 1)\,x + \frac{B_4}{\underline{|4}}\,.\;2\,(2^3 - 1)\,x^3$$

$$- \frac{B_6}{\underline{|6}}\,.\,2\,(2^5 - 1)\,x^5 + \ldots$$

and, since i cosech $ix = \text{cosec}\, x$,

$$\text{cosec}\, x = \frac{1}{x} + \frac{B_2}{\underline{|2}} \cdot 2\,(2-1)\,x + \frac{B_4}{\underline{|2}} \cdot 2\,(2^3-1)\,x^3$$
$$+ \frac{B_6}{\underline{|6}} \cdot 2\,(2^5-1)\,x^5 + \ldots$$

It will be afterwards shewn that the series for coth x and cot x and for cosech x and cosec x are convergent for values of x between $-\pi$ and π, and those for tanh x and tan x for values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

EXAMPLES.

[Throughout the following examples, Greek letters (θ, ϕ ...) denote the measures of angles in *radians*, other letters their measures in *right angles.*]

1) Using the notation of § 7, prove that, provided $n > -1$.

$$2^{\frac{n+1}{2}} \sin \frac{n+1}{2} = 1 + 1 \quad - 2 \quad - 3_n + 4 \quad + 5n - \ldots$$

$$2^{\frac{n+1}{2}} \cos \frac{n+1}{2} = 1 - 1 \quad - 2_n + 3 \quad + 4n - 5_n - 6_n + \ldots$$

(2) Expand $e^\theta \cos \theta$ and $e \quad \sin \quad \theta$ in ascending powers of θ, and show that the general terms of the expansions are $2^{\frac{n}{2}} \cos \frac{n}{2} \cdot \frac{\theta^n}{\underline{|n}}$ and $\frac{n}{2} \sin \frac{n}{2} \frac{\theta^2}{\underline{|n}}$ respectively.

(3) From the expansions of the previous question deduce those for $\cos \theta \,.\, \cosh \theta$, $\sin \theta \,.\, \cosh \theta$, $\cos \theta \,.\, \sinh \theta$, $\sin \theta \,.\, \sinh \theta$.

(4) From the series for $\sin \theta$, $\sinh \theta$, &c., verify the formulæ

A. $2 \sin \theta \cos \theta = \sin 2\theta$. B. $\cosh^2 \theta - \sinh^2 \theta = 1$.
C. $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$.

(5) Expand $\sin \theta - \theta \,.\, \cos \theta$ in powers of θ, and prove that, when θ vanishes, the limit of $(\sin \theta - \theta \cos \theta)/\theta^3 = \frac{1}{3}$.

Deduce that of $(\sin^2 \theta - \theta^2 \cos^2 \theta)/\theta^4$.

(6) Prove that $\lambda_e \frac{1 + i^u}{1 - i^u} = \log_e \cot \frac{u}{2} + i\frac{\pi}{2} + 2k\pi i$, and thence that, if $u > 4\lambda$ and $< 4\lambda + 2$,

$$\cos u + \frac{1}{3} \cos 3u + \frac{1}{5} \cos 5u + \ldots = \frac{1}{2} \log_e \left(\cot \frac{u}{2} \right),$$

$$\sin u + \frac{1}{3} \sin 3u + \frac{1}{5} \sin 5u + \ldots = \frac{\pi}{4}.$$

What, if $u > 4\lambda + 2$ and $< 4\lambda + 4$?

Illustrate the above by diagrams for particular values of u (*e.g.* $1^{\llcorner}$, $\frac{1}{2}^{\llcorner}$, $\frac{3}{2}^{\llcorner}$, $\frac{5}{3}^{\llcorner}$), and explain how discontinuity arises, when u passes through 2, 4, ... right angles.

(7) Prove that, with certain limitations, (—which determine—)

$$r\cos\theta + \frac{1}{3}r^3\cos 3\theta + \frac{1}{5}r^5\cos 5\theta + \ldots = \frac{1}{2}\operatorname{Tanh}^{-1}\frac{2r\cos\theta}{1+r^2},$$

$$r\sin\theta + \frac{1}{3}r^3\sin 3\theta + \frac{1}{5}r^5\sin 5\theta + \ldots = \frac{1}{2}\operatorname{Tan}^{-1}\frac{2r\sin\theta}{1-r^2}.$$

(8) Prove that, if θ lie between 0 and 2π,

$$\frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 4\theta + \frac{1}{6}\sin 6\theta + \ldots = \frac{\pi}{4} - \frac{\theta}{2}$$

also

$$\cos\theta - \frac{1}{2}\sin 2\theta - \frac{1}{3}\cos 3\theta + \frac{1}{4}\sin 4\theta + \frac{1}{5}\cos 5\theta - \ldots = \frac{\pi}{4} - \frac{\theta}{2}.$$

(9) If x is not greater than cosec a,

$$\operatorname{Tan}^{-1}(x + \cot a) = \frac{\pi}{2} - a + x\sin a\sin a - \frac{x^2}{2}\sin^2 a\sin 2a + \frac{x^3}{3}\sin^3 a\sin 3a + \ldots$$

(10) Prove that

$$\frac{\pi}{3} = 1 + \frac{1}{2}\cdot\frac{1}{3\times 2^2} + \frac{1\,.\,3}{2\,.\,4}\,\frac{1}{5\times 2^4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{7\times 2^6} + \ldots$$

(11) Prove that

$$\cos\theta + 2\cos 2\theta + \ldots + n\cos n\theta = \frac{\overline{n+1}\cos n\theta - n\cos\overline{n+1}\,\theta - 1}{4\sin^2\frac{\theta}{2}}$$

and

$$\sin\theta + 2\sin 2\theta + \ldots + n\sin n\theta = \frac{\overline{n+1}\sin n\theta - n\sin\overline{n+1}\,\theta}{4\sin\ \frac{\theta}{2}}$$

From the last show that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

(12) Prove that

$$\operatorname{Sin}^{-1}i = \frac{i}{\sqrt{2}}\left\{1 + \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{2^2} + \ldots\right\}.$$

(13) If $\sin\theta = n\sin(\theta + a)$, prove that, θ^2 being less than $\left(\frac{\pi}{2}\right)^2$,

$$\theta = n\sin a + \frac{1}{2}n^2\sin 2a + \frac{1}{3}n^3\sin 3a + \ldots,\ (n<1),$$

and

$$\theta = -a - \frac{1}{n}\sin a - \frac{1}{2}\frac{2}{n^2}\sin 2a - \ldots, \ (n>1).$$

Examine the case, where $n = 1$ and a is not 0; and illustrate the conclusion by geometry, regarding θ, $\theta + a$, as two angles of a triangle and n the ratio of the sides opposite to them.

(14) Prove the following summations to n terms:

A. $$\frac{1}{\cos a + \cos 3a} + \frac{1}{\cos a + \cos 5a} + \frac{1}{\cos a + \cos 7a} + \ldots$$
$$= \frac{\sin na}{\sin 2a \cos \overline{n+1}a}.$$

B. $$\frac{1}{\sin 3a - \sin a} + \frac{1}{\sin 5a - \sin a} + \frac{1}{\sin 7a - \sin a} + \ldots$$

$$= \frac{\sin na}{\sin 2a \,.\, \sin (n+1)a} \text{ or } \frac{\cos na}{\sin 2a \cos (\overline{n+1}a)},$$ as n is even or odd.

C. $\operatorname{cosec} a \,.\, \operatorname{cosec}(a+\beta) + \operatorname{cosec}(a+\beta)\operatorname{cosec}(a+2\beta)$

$$+ \operatorname{cosec}(a+2\beta) \,.\, \operatorname{cosec}(a+3\beta) + \ldots = \frac{\sin n\beta}{\sin\beta}\operatorname{cosec} a \operatorname{cosec}(a+n\beta).$$

D. $$\frac{1}{1 - \tanh a \tanh 2a} + \frac{1}{1 - \tanh 2a \tanh 4a} + \frac{1}{1 - \tanh 3a \tanh 6a} + \ldots$$
$$= \frac{\sinh na}{\sinh a}\cosh(\overline{n+1}\,a).$$

E. $\sin a \cos 2a + \sin 2a \,.\, \cos 3a + \sin 3a \,.\, \cos 2a + \ldots$

$$= \frac{(n+1)\cos 2a - \cos(\overline{n+1}\,2a) - n}{4 \sin a}.$$

Verify this by putting $a = 0$ after dividing both sides by $\sin a$.

F. $\operatorname{Sin} a \sin 2a + \sin 2a \,.\, \sin 3a + \sin 3a \,.\, \sin 4a + \ldots$

$$= \frac{(n+1)\sin 2a - \sin \overline{n+1}\,2a}{4 \sin a}.$$

Verify this by putting $a = 0$ after dividing both sides by $\sin^2 a$.

G. $$\frac{\sin a}{1 \,.\, \cos 2a} + \frac{\sin 3a}{\cos 2a \,.\, \cos 4a} + \frac{\sin 5a}{\cos 4a \,.\, \cos 6a} + \ldots$$
$$= \frac{\sin^2 na}{\sin a \,.\, \cos 2na}.$$

H. $\cos(a+\beta) + \cos(a+3\beta) + \cos(a+5\beta) + \ldots$

$$= \frac{\sin n\beta}{\sin\beta}\cos(a+n\beta).$$

I. $\sin^2 x \,.\, \sin 2x + \frac{1}{2}\sin^2 2x \,.\, \sin 4x + \frac{1}{2^2}\sin^2 4x \,.\, \sin 8x + \ldots$

$$= \frac{1}{2^{n+1}}(2^n \sin 2x - \sin 2^{n+1}x).$$

K. $\operatorname{Tan}^{-1} 2 + \operatorname{Tan}^{-1}\frac{4}{1+3\times 4} + \operatorname{Tan}^{-1}\frac{6}{1+7\times 8}$

$$+ \operatorname{Tan}^{-1}\frac{8}{1+15\times 16} + \ldots = \operatorname{Tan}^{-1} n(n+1).$$

L. $\operatorname{Tanh}^{-1} x + \operatorname{Tanh}^{-1}\frac{x}{1-1\,.\,2x^2} + \operatorname{Tanh}^{-1}\frac{x}{1-2\,.\,3x^2} + \ldots$

$$= \operatorname{Tanh}^{-1} nx.$$

M. $2\cos\frac{\theta}{2} + 2^2\cos\frac{\theta}{2}\,.\,\cos\frac{\theta}{2^2} + 2^3\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} + \ldots$

$$= \frac{\sin(2^n-1)\frac{\theta}{2^n}}{\sin\frac{\theta}{2^n}}.$$

(15) Express $\cot n\theta$ in terms of $\cot\theta$ (n a positive integer), and thence prove that

$$\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \cot\left(\theta + 2\frac{\pi}{n}\right) + \ldots \text{ to } n \text{ terms} = n\cot n\,\theta.$$

(16) $1 - 1_n 2\cos\theta \,.\, \cos\theta + 2_n \,.\, 2^2\cos^2\theta\cos 2\theta - \ldots = (-1)^n \cos 2n\theta.$

(17) $1 + n\cos\theta \,.\, \cos\theta + \ldots + \frac{\underline{|n+r-1}}{\underline{|n-1}\;\underline{|r}} \,.\, \cos^r\theta \,.\, \cos r\theta + \ldots$

$= (-1)^{\frac{n}{2}}\frac{\cos n\theta}{(\sin\theta)^n}$ or $(-1)^{\frac{n-1}{2}}\frac{\sin n\theta}{(\sin\theta)^n}$, according as n is even or odd.

(18) $\frac{\cos\theta}{\underline{|n}\,.\,\underline{|n+1}} + \frac{\cos 3\theta}{\underline{|n-1}\;\underline{|n+2}} + \frac{\cos 5\theta}{\underline{|n-2}\;\underline{|n+3}} + \ldots$

$$+ \frac{\cos(2n+1)\theta}{\underline{|2n+1}} = \frac{2^{2n}}{\underline{|2n+1}}(\cos\theta)^{2n+1}.$$

(19) $$1 + 3^n + 6_n + \dots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right).$$

(20) $$\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \dots \text{ (to } n \text{ terms)}$$
$$= \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta$$

and hence

$$\cot\theta = \frac{1}{\theta} - \frac{1}{2}\tan\frac{\theta}{2} - \frac{1}{2^2}\tan\frac{\theta}{2^2} - \dots$$

and

$$\frac{1}{4}\tan\frac{\pi}{4} + \frac{1}{8}\tan\frac{\pi}{8} + \frac{1}{16}\tan\frac{\pi}{16} + \dots = \frac{1}{\pi}.$$

(21) $$\cos a - \frac{\cos(a+2\beta)}{\underline{|3}} + \frac{\cos(a+4\beta)}{\underline{|5}} - \dots$$
$$= \cos(a-\beta)\cosh(\sin\beta)\sin(\cos\beta) - \sin(a-\beta)\sinh(\sin\beta)\cos(\cos\beta).$$

(22) $$\sin\theta\,.\,\frac{\sin\theta}{1} - \sin 2\theta\,.\,\frac{\sin^2\theta}{2} + \sin 3\theta\,\frac{\sin^3\theta}{3} - \dots$$
$$= \text{Cot}^{-1}\{\cot\theta + \operatorname{cosec}^2\theta\}.$$

(23) $$(1 \pm 1_n) - (2_n \pm 3_n)\,3 + (4_n \pm 5_n)\,3^3 - \dots$$
$$= 2^{\frac{n}{2}}\left\{\cos\frac{n\pi}{3} \pm \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3}\right\}.$$

(24) $$1 + \frac{\cos(a+\beta)}{\cos(a-\beta)} + \frac{\cos a\,.\,\cos(a+2\beta)}{\cos^2(a-\beta)} + \frac{\cos^2 a\,.\cos(a+3\beta)}{\cos^3(a-\beta)} + \dots$$
$$= 0, \text{ if } \cos a < \cos(a-\beta).$$

(25) $$2\sin\theta + \sin 3\theta + \cos\theta\sin 4\theta + \cos^2\theta\sin 5\theta + \dots = \operatorname{cosec}\theta.$$

(26) Find the coefficient of x^n in the expansion of $\{\log_e(1+x)\}^2$, and show that

$$\theta\,.\,\log_e\cos\frac{\theta}{2} = \sin 2\theta - \frac{2}{3}\left(1+\frac{1}{2}\right)\sin 3\theta + \frac{2}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)\sin 4\theta - \dots$$
$$+ (-1)^n\frac{2}{n}\,\Sigma_1^{n-1}\left(\frac{1}{r}\right).\sin n\theta + \dots.$$

(27) $$\sin^2 u - \frac{1}{2}\sin^2 2u + \frac{1}{3}\sin^2 3u - \dots = \frac{1}{2}\log(\sec x)$$

and

$$\cosh^2 u - \frac{1}{2}\cosh^2 2u + \frac{1}{3}\cosh^2 3u - \dots = \log(2\sqrt{\operatorname{sech} x}).$$

(28) $\cos u \,.\, \sin(u+n) + 1_n \cos(u+1)\sin(u+n-1)$
$+ 2_n \cos(u+2)\sin(u+n-2) + \dots$

$= (-1)^{\frac{n}{2}} 2^{n-1} \sin 2u$ or $(-1)^{\frac{n-1}{2}} 2^{n-1} \cos 2u$, as n is even or odd.

(29) From the formulæ, Ch. IV., § 8 (p. 95), deduce the following:

A. $\cos a + \cos\left(a + \frac{2\pi}{n}\right) + \cos\left(a + \frac{4\pi}{n}\right) + \dots$
$+ \cos\left(a + \overline{n-1}\,\frac{2\pi}{n}\right) = 0.$

B. $\sin a + \sin\left(a + \frac{2\pi}{n}\right) + \sin\left(a + \frac{4\pi}{n}\right) + \dots$
$+ \sin\left(a + \overline{n-1}\,\frac{2\pi}{n}\right) = 0.$

C. $\cos a \,.\, \cos\left(a + \frac{2\pi}{n}\right) \,.\, \cos\left(a + \frac{4\pi}{n}\right) \dots \cos\left(a + \overline{n-1}\,\frac{2\pi}{n}\right)$

$= \frac{1}{2^{n-1}} \cos na$ or $(-1)^{\frac{n}{2}} \frac{1}{2^{n-1}} \left\{1 - (-1)^{\frac{n}{2}} \cos na\right\}$, as n is odd or even.

D. $\sin a \,.\, \sin\left(a + \frac{2\pi}{n}\right) \,.\, \sin\left(a + \frac{4\pi}{n}\right) \dots \sin\left(a + \overline{n-1}\,\frac{2\pi}{n}\right)$

$= (-1)^{\frac{n-1}{2}} \frac{1}{2^{n-1}} \sin na$ or $(-1)^{\frac{n}{2}} \frac{1}{2^{n-1}} (1 - \cos na)$, as n is odd or even.

E. $\tan a + \tan\left(a + \frac{2\pi}{n}\right) + \dots + \tan\left(a + \overline{n-1}\,\frac{2\pi}{n}\right)$

$= n \tan a$ or $-n \cot\left(\frac{1}{2}na\right)$ or $n \tan\left(\frac{1}{2}na\right)$,

as n is of the form $2\lambda + 1$, 4λ, or $4\lambda + 2$.

F. $\sec a + \sec\left(a + \frac{2\pi}{n}\right) \dots + \sec\left(a + \overline{n-1}\,\frac{2\pi}{n}\right)$

$= (-1)^{\frac{n-1}{2}} n \sec na$ or 0, as n is odd or even.

(30) $1_n x + 4_n x^4 + 7_n x^7 + \dots$

$= \frac{1}{3}(1+x)^n - \frac{2}{3}(1 - x + x^2)^{\frac{n}{2}} \cos\left(n\theta + \frac{\pi}{3}\right),$

where $\theta = \text{Tan}^{-1}\frac{x\sqrt{3}}{2-x}$ and $x \not> 1$.

(31) $x - \frac{x^4}{4} + \frac{x^7}{7} - \ldots = \frac{1}{6}\log_e \frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}} \cdot \mathrm{Tan}^{-1}\frac{x\sqrt{3}}{2-x}, x \not> 1$

and hence $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \ldots = \frac{1}{3}\log_e 2 + \frac{\pi}{3\sqrt{3}}$.

(32) $1 - \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \ldots = \frac{1}{2\sqrt{2}}\left\{\log_e(\sqrt{2}+1) + \frac{\pi}{2}\right\}$

$$= \frac{1}{\sqrt{2}}\left\{\mathrm{Sinh}^{-1}1 + \mathrm{Tan}^{-1}1\right\}.$$

(33) $\frac{1}{e^x+1} = \frac{1}{2} - \frac{B_2}{\underline{|2}}(2^2-1)x + \frac{B_4}{\underline{|4}}(2^4-1)x^3 - \frac{B_6}{\underline{|6}}(2^6-1)x^5 + \ldots$

and deduce the series for tan x and tanh x.

(34) If $\tan x = C_2 x + C_4 x^3 + C_6 x^5 + \ldots,$

$$\left(\text{so that } C_{2n} = B_{2n}\frac{2^{2n}(2^{2n}-1)}{\underline{|2n}}\right),$$

prove, from the formula $\sin x = \tan x \cos x$, that

$$C_{2n} = \frac{C_{2n-2}}{\underline{|2}} - \frac{C_{2n-4}}{\underline{|4}} + \ldots + (-1)^n \frac{C_2}{\underline{|2n-2}} - \frac{(-1)^n}{\underline{|2n-1}},$$

and thence calculate the values of C_2, C_4 … to C_{10}.

(35) Prove that the roots of the equation

$$x^n + 1_n \cos a \,.\, x^{n-1} + 2_n \cos 2a \,.\, x^{n-2} + \ldots + \cos na = 0$$

are the n values of the expression $\sin a \cot \frac{2\lambda+1}{n} - \cos a$ for n consecutive integral values of λ.

Also that the roots of

$$0 \,.\, x^n + 1_n \sin a \,.\, x^{n-1} + 2_n \sin 2a \,.\, x^{n-2} + \ldots + \sin na = 0,$$

are the values of $\sin a \cot \frac{2\lambda}{n} - \cos a$.

CHAPTER X

SERIES OF FACTORS

The circular and ex-circular functions, besides being capable of expression by infinite series of terms proceeding by powers of the measure of the angle, as shown in the last chapter, can also be expressed in terms of such measure by products of an infinite number of factors, and by the sums of infinite series of simple fractions. It is the object of this and the following chapter to investigate these series and deduce from them some important analytical results. It is necessary to premise certain propositions as to the convergency of Infinite Products.

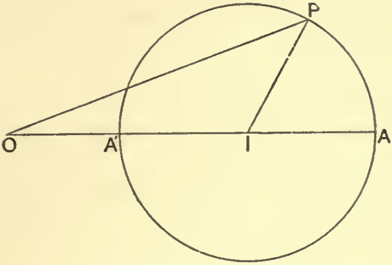
1. *Infinite Products. Convergence or Divergence.*

Let $z_1, z_2, \ldots z_n$ be a series of vectors, whose tensors are each less than 1, then the product $(1+z_1)\ (1+z_2)\ \ldots\ (1+z_n)$, which we shall denote as $\Pi_1^n(1+z_p)$, may, when n is increased without limit, increase without limit, in which case it is said to be *divergent;* or it may have a definite finite limit, in which case it is said to be *convergent;* or it may be finite, but have no definite limit: or it may diminish without limit and ultimately vanish.

Now, if $z = ri^u$, $1+z = 1+ri^u$ and mod. $(1+z)$ lies between the limits $1+r$ and $1-r$, or as it may be expressed,

$$\text{mod.}\ (1+z) \begin{matrix} < 1+r \\ > 1-r \end{matrix}.$$

This is at once evident from the figure in which $OI = 1$, $\overline{IP} = z$ and therefore $\overline{OP} = 1 + z$ and OP, the tensor of $1 + z$, is between the limits OA, OA' or $1 + r$, $1 - r$.



Hence

$$\text{mod.}\,(1 + z_1)(1 + z_2)\ldots(1 + z_n) \begin{matrix} < (1 + r_1)(1 + r_2)\ldots(1 + r_n) \\ > (1 - r_1)(1 - r_2)\ldots(1 - r_n) \end{matrix}.$$

Now ultimately, when n is infinite, the product

$$(1 + r_1)(1 + r_2)\ldots(1 + r_n) \quad \text{or} \quad \Pi_1^n (1 + r_p),$$

is evidently greater than 1 and may be infinite, while

$$(1 - r_1)\ldots(1 - r_n) \quad \text{or} \quad \Pi_1^n (1 - r_p)$$

is evidently less than 1 and may be zero. Also since

$$(1 + r)(1 - r) = 1 - r^2 < 1,$$

if $\Pi_1^\infty (1 + r_p)$ is infinite $\Pi_1^\infty (1 - r_p) = 0$, and if $\Pi_1^\infty (1 - r_p)$ is finite, $\Pi_1^\infty (1 + r_p)$ is finite also.

Also, developing the product $\Pi_1^n (1 + r_p)$ by multiplying out its factors, the resulting terms are all positive, and therefore omitting those which involve products of the r's,

$$\Pi_1^n (1 + r_p) > 1 + r_1 + r_2 + \ldots + r_n,$$

or (as it may be denoted) $1 + \Sigma_1^n (r_p)$.

Hence if the series $\Sigma_1^\infty(r_p)$ is divergent, $\Pi_1^\infty(1+r_p)=\infty$ and therefore $\Pi_1^\infty(1-r_p)=0$. In this case mod. $\Pi_1^\infty(1+z_p)$ may be infinite, finite, or zero, according to the law by which z_p depends on p, and in the absence of any general test, the convergence or divergence of the infinite product must be investigated in each particular case.

Again,

$$(1-r_m)(1-r_{m+1})=1-r_m-r_{m+1}+r_m r_{m+1}>1-r_m-r_{m+1},$$

$$(1-r_m)(1-r_{m+1})(1-r_{m+2})>(1-r_m-r_{m+1})(1-r_{m+2})$$
$$>1-r_m-r_{m+1}-r_{m+2},$$

and generally

$$(1-r_m)(1-r_{m+1})\ldots(1-r_{m+n})>1-r_m-r_{m+1}-\ldots-r_{m+n}$$

or

$$\Pi_m^{m+n}(1-r_p)>1-\Sigma_m^{m+n}(r_p).$$

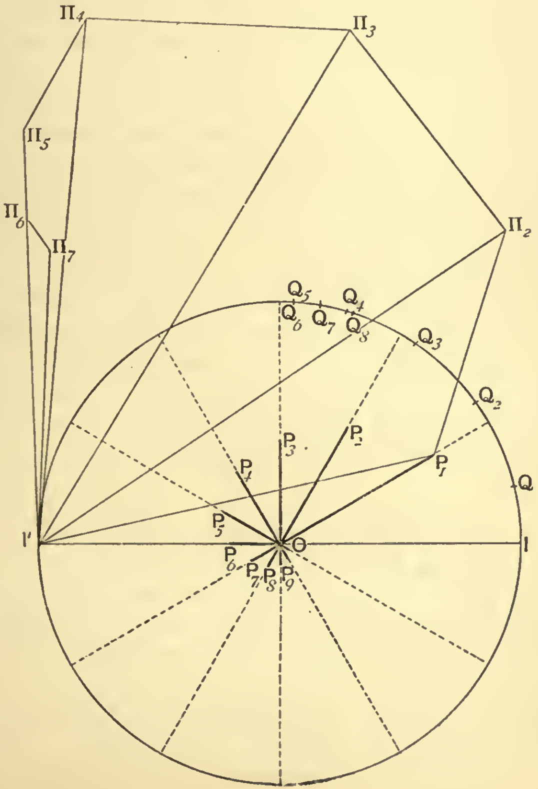
Now if $\Sigma_1^\infty(r_p)$ is convergent or has a definite finite limit, a finite value of m can be found such that $\Sigma_1^m(r_p)$ differs from that limit by less than 1, so that $\Sigma_{m+1}^\infty(r_p)$ is a finite quantity less than 1, which may be made as small as we please by taking m sufficiently great: hence $\Pi_{m+1}^\infty(1-r_p)$, though a finite quantity less than 1, may be made as near as we please to 1 by taking m sufficiently great, and therefore, $\Pi_1^m(1-r_p)$ being a finite quantity less than 1, $\Pi_1^\infty(1-r_p)$ is finite and has a definite limit less than 1 and greater than 0. Hence also $\Pi_1^\infty(1+r_p)$ is finite and has a definite limit greater than 1.

It follows therefore that mod. $\Pi_1^\infty(1+z_p)$ is finite and between the finite limits of $\Pi_1^\infty(1-r_p)$ and $\Pi_1^\infty(1+r_p)$, if the series $\Sigma_1^\infty(r_p)$ is convergent; but may be infinite, finite, or zero, if $\Sigma_1^\infty(r_p)$ is divergent.

In order to prove that when mod. $\Pi_1^\infty(1+z_p)$ is a definite finite quantity, $\Pi_1^\infty(1+z_p)$ converges to a determinate limit, it is necessary further to show that, v denoting the inclination of the vector $1+ri^u$ so that $v = \text{Tan}\,\dfrac{r\sin u}{1+r\cos u}$, $v_1+v_2+v_3+\dots$ or $\Sigma_1^\infty(v_p)$ converges to a definite finite limit: since

$$\Pi_1^\infty(1+z_p) = \text{mod.}\,\Pi_1^\infty(1+z_p)\,.\,i^{v_1+v_2+v_3+\dots}.$$

Ex. Take as an example the product $(1+z)(1+z^2)(1+z^3)\dots$ *ad inf.* or $\Pi_1^\infty(1+r^p i^{pu})$, and in the figure (drawn for the case



where $r = \frac{3}{4}$, $u = \frac{1}{3}\llcorner$ or 30°) let $\overline{OP}_1$, $\overline{OP}_2$, $\overline{OP}_3$ . . . represent the vectors z, z^2, z^3, . . ., all within the unit circle, centre O, then $\Pi_1^\infty(1 + r^p i^{pu})$ is the product of the vectors $\overline{I'P}_1$, $\overline{I'P}_2$, $\overline{I'P}_3$ . . .

Then since $r + r^2 + r^3 + \ldots$ has the finite limit $\frac{r}{1-r}$, or the sum of the tensors of $\overline{OP}_1$, $\overline{OP}_2$ . . . is finite, the modulus of the infinite product $\Pi_1^\infty(1 + r^p i^{pu})$ or the tensor of the product $\overline{I'P}_1 . \overline{I'P}_2 . \overline{I'P}_3$ . . . has a definite finite limit. Also the angles v_1, v_2, . . . or $\overline{OI'P}_1$, $\overline{OI'P}_2$, . . . are respectively

$$\operatorname{Tan}^{-1}\frac{r\sin u}{1 + r\cos u},\ \operatorname{Tan}^{-1}\frac{r^2\sin 2u}{1 + r^2\cos 2u},\ \operatorname{Tan}^{-1}\frac{r^3\sin 3u}{1 + r^3\cos 3u}\ \ldots,$$

and since the points P_{n+1}, P_{n+2}, . . . all lie within the circle, centre O and radius OP_n, the angles v_{n+1}, v_{n+2}, . . . v_{n+m} all lie between $\sin^{-1}OP_n$ and $-\sin^{-1}OP_n$ so that their sum lies between $m . \sin^{-1}(r^n)$ and $-m\sin^{-}(r^n)$, which may be made as small as we please by taking n sufficiently great, so that $v_1 + v_2 + v_3 + \ldots$ has a definite finite limit. Hence the infinite product $\Pi_1^\infty(1 + r^p i^{pu})$ is a definite finite vector.

In the figure IQ_1, Q_1Q_2, Q_2Q_3 . . . are arcs subtending angles at O equal to v_1, v_2, v_3 . . ., and it is obvious that their sum is finite and definite. The limiting vector might be approximated to, by geometrical construction, by describing on $I'P$, a triangle $I'P_1\Pi_2$ similar to $I'OP_2$, and then on $I'\Pi_2$ a triangle $I'\Pi_2\Pi_3$ similar to $I'OP_3$ and so on continually, as in the figure, where $\overline{I'P}_1$, $\overline{I'\Pi}_2$, $\overline{I'\Pi}_3$ . . . are the successive products of 1, 2, 3 . . . factors.

If $r = 1$, the vectors $\overline{OP}_1$ $\overline{OP}_2$ . . . are radii of the unit circle and $\overline{I'P}_1$, $\overline{I'P}_2$ . . . chords of the same, and their product, if u is commensurable with a right angle, resolves itself into periods of products repeated over and over again *ad inf.*, and if for any one period the product is finite, which is the case if no P coincides with I', the infinite product will have a tensor 0 or ∞, as the product of the tensors of the period is less or greater than 1. If one of the P's coincided with I', there would be a zero factor in each period, and the infinite product would be 0.

2. *Resolution of $z^n - a^n$ into factors.*

If n is a positive integer, by algebraical division

$$\frac{z^n - a^n}{z-a} = z^{n-1} + az^{n-2} + a^2z^{n-3} + \ldots + a^{n-2}z + a^{n-1},$$

and the quotient becomes na^{n-1}, when $z = a$.

But $a^n = i^{4\lambda}a^n = (i^{\frac{4\lambda}{n}}a)^n$, λ being any positive or negative integer, hence $z^n - a^n$ is divisible by $x - i^{\frac{4\lambda}{n}} a$. Now $i^{\frac{4\lambda}{n}}$ has n, and only n, different values corresponding to any n consecutive integral values of λ, which may be most conveniently taken in the symmetrical sequence

$$-\frac{n-1}{2},\ -\frac{n-b}{2}, \ldots -3,\ -2,\ -1, 0, 1, 2 \ldots \frac{n-3}{2}, \frac{n-1}{2},$$

when n is odd, or

$$-\left(\frac{n}{2}-1\right),\ -\left(\frac{n}{2}-2\right) \ldots -2,\ -1, 0, 1, 2 \ldots \left(\frac{n}{2}-1\right), \frac{n}{2},$$

when n is even, and $z^n - a^n$ cannot have more than n binomial factors, therefore

$$z^n - a^n = (z-a)\left\{\begin{array}{c}(z-i^{\frac{4}{n}}a).(z-i^{\frac{8}{n}}a)\ldots(z-i^{\frac{2n-2}{n}}a)\\ (z-i^{-\frac{4}{n}}a).(z-i^{-\frac{8}{n}}a)\ldots(z-i^{-\frac{2n-2}{n}}a)\end{array}\right\}..A_1$$

if n is odd, or

$$=(z-a)\left\{\begin{array}{c}(z-i^{\frac{4}{n}}a)(z-i^{\frac{4}{n}}a)\ldots(z-i^{\frac{2n-4}{n}}a)\\ (z-i^{-\frac{4}{n}}a)(z-i^{-\frac{8}{n}}a)\ldots(z-i^{-\frac{2n-4}{n}}a)\end{array}\right\}(z-i^2a)..A_2$$

if n is even.

But $$(z-i^{\frac{4\lambda}{n}}a)(z-i^{-\frac{4\lambda}{n}}a) = z^2 - (i^{\frac{4\lambda}{n}} + i^{-\frac{4\lambda}{n}})az + a^2$$

$$= z^2 - 2az\cos\frac{4\lambda}{n} + a^2,$$

hence the series above may be written in the shape

$$z^n - a^n = (z-a)\left(z^2 - 2az\cos\frac{4}{n} + a^2\right)\left(z^2 - 2az\cos\frac{8}{n} + a^2\right)$$

$$\ldots\left(z^2 - 2az\cos\frac{2n-2}{n} + a^2\right) \ldots B_1$$

if n is odd, or

$$z^n - a^n = (z-a)(z+a)\left(z^2 - 2az\cos\frac{4}{n} + a^2\right)$$

$$\ldots\left(z^2 - 2az\cos\frac{2n-4}{n} + a^2\right) \ldots B_2$$

if n is even; in which, if z, a are scalar, all the factors are scalar.

In these formulae, if we write $i^{\frac{2}{n}}a$ in the place of a and therefore $-a^n$ for a^n,

$$z^n + a^n = \left\{\begin{matrix} (z - i^{\frac{2}{n}}a)(z - i^{\frac{6}{n}}a) \ldots (z - i^{\frac{2n-4}{n}}a) \\ (z - i^{-\frac{2}{n}}a)(z - i^{-\frac{6}{n}}a) \ldots (z - i^{-\frac{2n-4}{n}}a) \end{matrix}\right\}(z+a),\ n \text{ odd} \ldots$$

$$= (z+a)\left(z^2 - 2az\cos\frac{2}{n} + a^2\right)\left(z^2 - 2az\cos\frac{6}{n} + a^2\right)$$

$$\ldots\left(z - 2az\cos\frac{2n-4}{n} + a^2\right) \ldots C_1$$

and, since $\quad z - i^2 . i^{\frac{2}{n}}a = z - i^4 . i^{\frac{2}{n}-2}a = z - i^{-\frac{2n-2}{n}}a,$

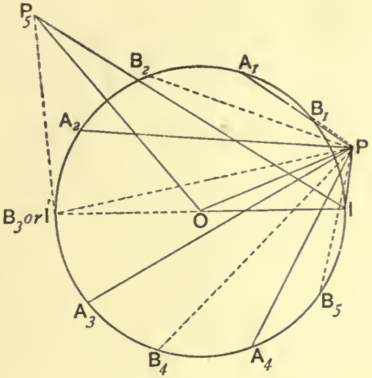
$$z^n + a^n = \left\{\begin{matrix} (z - i^{\frac{2}{n}}a)(z - i^{\frac{6}{n}}a) \ldots (z - i^{\frac{2n-2}{n}}a) \\ (z - i^{-\frac{2}{n}}a)(z - i^{-\frac{6}{n}}a) \ldots (z - i^{-\frac{2n-2}{n}}a) \end{matrix}\right\},\ n \text{ even} \ldots$$

$$= \left(z^2 - 2az\cos\frac{2}{n} + a^2\right)\left(z^2 - 2az\cos\frac{6}{n} + a^2\right)$$

$$\ldots\left(z^2 - 2az\cos\frac{2n-2}{n} + a^2\right) \ldots C_2.$$

3. *Geometrical Interpretation. Cotes's and De Moivre's Properties of the Circle.*

Let the circumference of the unit circle, centre O, be divided into n equal parts (in the figure $n=5$) at the points $I, A_1, A_2, \ldots A_{n-1}$ and let $\overline{OP}=z$. We will take the case, where $a=\overline{OI}$ or 1.



Then if OP_n make with OI the angle $IOP_n=n\,.\,IOP$ and $OP_n=OP^n$,

$$z^n-1=(\overline{OP})^n-(\overline{OI})^n=\overline{OP}_n-\overline{OI}=IP_n.$$

Also

$$z-1=\overline{OP}-\overline{OI}=IP,\ z-i^{\frac{4}{n}}=\overline{OP}-\overline{OA_1}=\overline{A_1P},\ z-i^{\frac{8}{n}}=\overline{A_2P},$$

and so on, hence, by the formulae A_1 or A_2 of the last Article,

$$\overline{IP}_n=\overline{OP}^n-\overline{OI}^n=\overline{IP}\,.\,\overline{A_1P}\,.\,\overline{A_2P}\,\ldots\,\overline{A_{n-1}P},$$

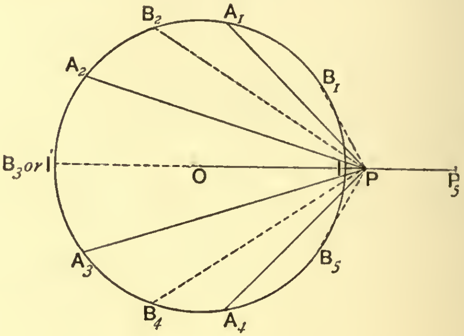
so that both the tensor IP_n of $\overline{OP}^n-\overline{OI}^n$ is equal to the product of the tensors of $\overline{IP}, \overline{A_1P} \ldots \overline{A_{n-1}P}$ and the inclination of IP_n to OI is the sum of the inclinations of the same vectors to OI.

If P, and therefore P_n, are on the line OI, $\overline{OP}$ and $\overline{OP}^n$ or $\overline{OP}_n$ are scalar, and in that case the sum of the inclinations of $\overline{IP}$,

$\overline{A_1P} \dots \overline{A_{n-1}P}$ is zero, as is obvious from symmetry, and we have the scalar relation

$$IP_n = OP^n - OI^n = IP \,.\, A_1P \,.\, A_2P \dots A_{n-1}P,$$

a property, which is known as Cotes's Theorem.



If $z=1$, P coincides with I, and IP, IP_n both vanish, but when z becomes 1,

$$\text{lt.}\ \frac{IP_n}{IP} = \text{lt.}\ \frac{z^n-1}{z-1} = n,$$

and we have $$A_1I \,.\, A_2I \dots A_{n-1}I = n,$$

or $$\text{chd.}\ \frac{4}{n} \,.\, \text{chd.}\ \frac{8}{n} \,.\, \text{chd.}\ \frac{12}{n} \dots \text{chd.}\ \frac{4n-4}{n} = n,$$

or $$2^{n-1} \sin\frac{2}{n} \,.\, \sin\frac{4}{n} \,.\, \sin\frac{6}{n} \dots \sin\frac{2n-2}{n} = n.$$

If B_1, B_2, $B_3 \dots B_n$ are the middle points of the arcs IA_1, A_1A_2, A_2A_3, $\dots$ $A_{n-1}I$, we obtain in like manner from the formula C_1 or C_2,

$$\overline{IP}_n = \overline{OP}^n + \overline{OI}^n = \overline{B_1P} \,.\, \overline{B_2P} \dots \overline{B_nP}.$$

When $\overline{OP}$ is scalar, P and P_n are on OI, and we have the scalar relation known as Cotes's Second Theorem.

If P coincides with I, $I'P=2$, and this theorem gives

$$B_1I \,.\, B_2I \,.\,.\,.\, B_nI=2,$$

or
$$\text{chd.}\,\frac{2}{n}\,\text{chd.}\,\frac{6}{n}\,.\,.\,.\,\text{chd.}\,\frac{4n-2}{n}=2,$$

or
$$2^{n-1}\sin\frac{1}{n}\,.\,\sin\frac{3}{n}\,.\,.\,.\,\sin\frac{2n-1}{n}=1.$$

Since $(\text{tensor}\ \overline{IP_n})^2=IP_n{}^2=OP_n{}^2-2OI\,.\,OP_n\cos IOP_n+OI^2$

$$=OP^{2n}-2OP^n\cos nu+1,$$

for
$$OP_n=OP^n \text{ and } IOP_n=n\,.\,IOP$$

and
$$(\text{tensor}\ \overline{I'P_n})^2=I'P_n{}^2=OP_n{}^2+2OI\,.\,OP_n\cos IOP_n+OI^n$$

$$=OP^{2n}+2OP^n\cos nu+1$$

we have $OP^{2n}-2OP^n\cos nu+1=IP^2\,.\,A_1P^2\,.\,A_2P^2\,.\,.\,.\,A_{n-1}P^2$

$$OP^{2n}+2OP^n\cos nu+1=B_1P^2\,.\,B_2P^2\,.\,B_3P^2\,.\,.\,.\,B_nP^2,$$

results which are known as De Moivre's Property of the Circle.

4. *Resolution of* $z^{2n}-2a^nz^n\cos u+a^{2n}$ *into factors.*

Since
$$2\cos u=i^u+i^{-u},$$

$$z^{2n}-2a^nz^n\cos u+a^{2n}=(z^n-i^ua^n)(z^n-i^{-u}a^n)$$

$$=\left\{z^n-\left(i^{\frac{4\lambda+u}{n}}a\right)^n\right\}\left\{z^n-\left(i^{-\frac{4\lambda+u}{n}}a\right)^n\right\},$$

λ being any integer.

Each of these factors may, as in the last article, be resolved into n factors of the form $z-i^{\frac{4\lambda+u}{n}}a$, $z-i^{-\frac{4\lambda+u}{n}}a$ respectively by giving λ any n consecutive integral values, and for the same value of λ in each,

$$\left(z-i^{\frac{4\lambda+u}{n}}a\right)\left(z-i^{-\frac{4\lambda+u}{n}}a\right)=z^2-2az\cos\frac{4\lambda+u}{n}+a^2.$$

Hence, putting λ successively $= 0, 1, 2 \ldots, n-1$,

$$z^{2n} - 2a^n z^n \cos u + a^{2n} = \left(z^2 - 2az\cos\frac{u}{n} + a^2\right)\left(z^2 - 2az\cos\frac{4+u}{n} + a^2\right)$$

$$\ldots \left(z^2 - 2az\cos\frac{4\overline{n-1}+u}{n} + a^2\right).$$

And, putting $u+2$ for u, and therefore $-\cos u$ for $\cos u$,

$$z^{2n} + 2a^n z^n \cos u + a^{2n} = \left(z^2 - 2az\cos\frac{2+u}{n} + a^2\right)$$

$$\left(z^2 - 2az\cos\frac{6+u}{n} + a^2\right) \ldots \left(z^2 - 2az\cos\frac{2(2n-1)+u}{n} + a^2\right).$$

The geometrical interpretation of these formulæ gives De Moivre's properties of the circle.

5. *Factor Series for sine and cosine.*

In the formula B_1, B_2 and C_1, C_2 of § 2, put $z = i^u$ and $a = i^{-u}$: then, since

$$z - a = 2i\sin u,\ z + a = 2\cos u,\ z^n - a^n = 2i\sin nu,\ z^n + a^n = 2\cos nu,$$

$$z^n - 2az\cos\frac{4\lambda}{n} + a^2 = i^{2u} + i^{-2u} - 2\cos\frac{4\lambda}{n} = 2\left(\cos 2u - \cos\frac{4\lambda}{n}\right)$$

$$= 2^2\left(\sin^2\frac{2\lambda}{n} - \sin^2 u\right),$$

$$z^n - 2az\cos\frac{4\lambda+2}{n} + a^2 = i^{2u} + i^{-2u} - 2\cos\frac{4\lambda+2}{n}$$

$$= 2\left(\cos 2u - \cos\frac{4\lambda+2}{n}\right) = 2^2\left(\sin^2\frac{2\lambda+1}{n} - \sin^2 u\right),$$

we have

$$2i\,.\sin nu = 2i\,.\sin u\,.\,2^2\left(\sin^2\frac{2}{n} - \sin^2 u\right).\,2^2\left(\sin^2\frac{4}{n} - \sin^2 u\right)$$

$$\begin{cases} 2^2\left(\sin^2\dfrac{n-1}{n} - \sin^2 u\right),\ n \text{ odd} \\[2ex] 2^2\left(\sin^2\dfrac{n-2}{n} - \sin^2 u\right).\,2\cos u,\ n \text{ even,} \end{cases}$$

$$2\cos nu = 2^2\left(\sin^2\frac{1}{n}-\sin^2 u\right).\,2^2\left(\sin^2\frac{3}{n}-\sin^2 u\right)$$

$$\ldots\begin{cases}2^2\left(\sin^2\dfrac{n-2}{n}-\sin^2 u\right).\,2\cos u,\ n \text{ odd}\\[2ex] 2^2\left(\sin^2\dfrac{n-1}{n}-\sin^2 u\right),\ n \text{ even}\end{cases}$$

whence, writing $\frac{u}{n}$ for u,

$$\frac{\sin u}{\sin\frac{u}{n}} = 2^{n-1}.\left(\sin^2\frac{2}{n}-\sin^2\frac{u}{n}\right)\left(\sin^2\frac{4}{n}-\sin^2\frac{u}{n}\right)$$

$$\ldots\begin{cases}\left(\sin^2\dfrac{n-1}{n}-\sin^2\dfrac{u}{n}\right),\ n \text{ odd}\\[2ex] \left(\sin^2\dfrac{n-2}{n}-\sin^2\dfrac{u}{n}\right)\cos\dfrac{u}{n},\ n \text{ even}\end{cases},$$

$$\cos u = 2^{n-1}\left(\sin^2\frac{1}{n}-\sin^2\frac{u}{n}\right)\left(\sin^2\frac{3}{n}-\sin^2\frac{u}{n}\right)$$

$$\ldots\begin{cases}\left(\sin^2\dfrac{n-2}{n}-\sin^2\dfrac{u}{n}\right)\cos\dfrac{u}{n},\ n \text{ odd}\\[2ex] \left(\sin^2\dfrac{n-1}{n}-\sin^2\dfrac{u}{n}\right),\ n \text{ even}\end{cases}.$$

Making $u=0$ in these formulæ, since $\dfrac{\sin u}{\sin\frac{u}{n}} = n$ ultimately, when $u=0$,

$$n = 2^{n-1}.\sin^2\frac{2}{n}.\sin^2\frac{4}{n}\ldots\begin{cases}\sin^2\dfrac{n-1}{n},\ n \text{ odd}\\[2ex] \sin^2\dfrac{n-2}{n},\ n \text{ even}\end{cases},$$

$$1 = 2^{n-1}.\sin^2\frac{1}{n}.\sin^2\frac{3}{n}\ldots\begin{cases}\sin^2\dfrac{n-2}{n},\ n \text{ odd}\\[2ex] \sin^2\dfrac{n-1}{n},\ n \text{ even}\end{cases}.$$

whence

$$\frac{\sin u}{n\sin\frac{u}{n}}=\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{2}{n}}\right)\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{4}{n}}\right)\dots\left\{\begin{array}{l}\left(1-\dfrac{\sin^2\frac{u}{n}}{\sin^2\frac{n-1}{n}}\right),\ n \text{ odd}\\[2ex] \left(1-\dfrac{\sin^2\frac{u}{n}}{\sin^2\frac{n-2}{n}}\right)\cos\dfrac{u}{n},\ n\text{ even,}\end{array}\right.$$

$$\cos u=\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{1}{n}}\right)\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{3}{n}}\right)\dots\left\{\begin{array}{l}\left(1-\dfrac{\sin^2\frac{u}{n}}{\sin^2\frac{n-2}{n}}\right)\cos\dfrac{u}{n},\ n \text{ odd.}\\[2ex] \left(1-\dfrac{\sin^2\frac{u}{n}}{\sin^2\frac{n-1}{n}}\right),\ n\text{ even}\end{array}\right.$$

Now, suppose n to increase without limit, then

$$\text{lt. } n\sin\frac{u}{n}=u\,\text{lt.}\frac{\sin\frac{u}{n}}{\frac{u}{n}}=u\,.\,\frac{\pi}{2},$$

and

$$\text{lt.}\frac{\sin\frac{u}{n}}{\sin\frac{p}{n}}=\frac{u}{p},$$

so that, *if we can prove* that the products on the right-hand side of these equations are convergent, when the number of factors becomes infinite, we have

$$\frac{\sin u}{u}=\frac{\pi}{2}\left(1-\frac{u^2}{2^2}\right)\left(1-\frac{u^2}{4^2}\right)\left(1-\frac{u^2}{6^2}\right)\dots\ \textit{ad inf.}$$

$$\cos u=\left(1-\frac{u^2}{1^2}\right)\left(1-\frac{u^2}{3^2}\right)\left(1-\frac{u^2}{5^2}\right)\dots\ \textit{ad inf.}$$

Let m be an even number less than n (n being odd) but greater than u, so that $\sin\frac{u}{n}\Big/\sin\frac{m}{n}$ is less than 1, and let R denote the product of all the factors in the expression for $\sin u/n \sin\frac{u}{n}$ after the factor $1-\dfrac{\sin^2\frac{u}{n}}{\sin^2\frac{m}{u}}$, so that

$$R=\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{m+2}{n}}\right)\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{m+4}{n}}\right)\cdots\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{n-1}{n}}\right).$$

Then * R is less than 1, but greater than

$$1-\sin^2\frac{u}{n}\left(\operatorname{cosec}^2\frac{m+2}{n}+\operatorname{cosec}^2\frac{m+4}{n}+\ldots+\operatorname{cosec}^2\frac{n-1}{n}\right).$$

Now $\sin u/u$, as u increases from 0 to 1, continually decreases † from $\pi/2$ to 1, so that for any acute angle measured by u, $\sin u<\frac{\pi}{2}u$ and $>u$, and therefore $\operatorname{cosec} u>\frac{2}{\pi}\frac{1}{u}$ and $<\frac{1}{u}$, so that, substituting for the cosecants in the expression above the reciprocals of their arguments, still more is

$$R>1-\sin^2\frac{u}{n}\left\{\frac{n^2}{(m+2)^2}+\frac{n^2}{(m+4)^2}+\ldots+\frac{n^2}{(n-1)^2}\right\},$$

and, since $\sin\frac{u}{n}<\frac{\pi}{2}\cdot\frac{u}{n}$, still more is

$$R>1-\frac{\pi^2}{4}u^2\cdot\left\{\frac{1}{(m+2)^2}+\frac{1}{(m+4)^2}+\ldots+\frac{1}{(n-1)^2}\right\},$$

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* This proof is given by Schlömilch as due to Prof. Schröter (*Compendium d. Höheren Analysis*, 1874).

† See p. 149, Ex. 9.

but

$$\frac{1}{(m+2)^2}+\frac{1}{(m+4)^2}+\ldots+\frac{1}{(n-1)^2}<\frac{1}{m(m+2)}+\frac{1}{(m+2)(m+4)}$$
$$+\ldots+\frac{1}{(n-3)(n-1)}$$
$$<\frac{1}{2}\left\{\frac{1}{m}-\frac{1}{m+2}+\frac{1}{m+2}-\frac{1}{m+4}+\ldots+\frac{1}{n-3}-\frac{1}{n-1}\right\}$$
$$<\frac{1}{2}\left(\frac{1}{m}-\frac{1}{n-1}\right)$$
$$<\frac{1}{2m},$$

therefore $$R>1-\frac{\pi^2}{8}\frac{u^2}{m},$$

and we may write

$$R=1-\epsilon\frac{\pi^2}{8}\frac{u^2}{m},$$

where ϵ is some quantity between 0 and 1.

Hence

$$\frac{\sin u}{n\sin\frac{u}{n}}=\left(1-\frac{\sin^2\frac{4}{n}}{\sin^2\frac{2}{n}}\right)\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{4}{n}}\right)\ldots\left(1-\frac{\sin^2\frac{u}{n}}{\sin^2\frac{m}{n}}\right)\left(1-\epsilon\frac{\pi^2}{8}\frac{u^2}{m}\right),$$

and, if ϵ_1 denote the value of ϵ, when n is increased without limit

$$\frac{\sin u}{u}=\frac{\pi}{2}\left(1-\frac{u^2}{2^2}\right)\left(1-\frac{u^2}{4^2}\right)\ldots\left(1-\frac{u^2}{m^2}\right)\left(1-\epsilon_1\frac{\pi^2}{8}\cdot\frac{u^2}{m}\right),$$

so that, the last factor becoming 1 when n is increased without limit,

$$\frac{\sin u}{u}=\frac{\pi}{2}\left(1-\frac{u^2}{2^2}\right)\left(1-\frac{u^2}{4^2}\right)\left(1-\frac{u^2}{6^2}\right)\ldots\ \textit{ad inf.}$$

as we were led to expect above.

It is plain that the same reasoning would apply to the series for $\cos u$, the only difference being that m must be taken to be an odd number, and also that the difference of the cases, when n is even and when odd, disappears, when n is infinite, since then

$$\cos \frac{u}{n} = 1.$$

If θ denote the circular measure of the angle, since $\theta^{\odot} = \frac{\pi}{2} u^{\llcorner}$, the formulæ become

$$\frac{\sin \theta}{\theta} = \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \ldots \textit{ad inf.}$$

$$\cos \theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4^2\theta^2}{5^2\pi^2}\right) \ldots \textit{ad inf.}$$

This last is the form, in which these series are usually written, as, in analysis generally, it is most convenient to use circular measure.

These formulæ have been proved on the supposition that u is scalar, but they are also valid, when u is a complex number or vector. For the formulæ above for a finite value of n are true without restriction as to the value of u from the laws of Algebra, or the principle of the permanence of equivalent forms, and it is only necessary to show that the deduction from them, when n becomes infinite, also holds good.

Let $u = x + iy$, then

$$\sin \frac{u}{n} = \sin \frac{x}{n} . \cos \frac{iy}{n} + \cos \frac{x}{n} . \sin \left(\frac{iy}{n}\right) = \sin \frac{x}{n} \cosh \frac{y}{n} + i \cos \frac{x}{n} \sinh \frac{y}{n},$$

and if ρ be the modulus of $\sin \frac{u}{n}$,

$$\rho^2 = \sin^2 \frac{x}{n} \cosh^2 \frac{y}{n} + \cos^2 \frac{x}{n} \sinh^2 \frac{y}{n} = \sin^2 \frac{x}{n} + \sinh^2 \frac{y}{n},$$

and as we have seen in Art. 1, the product is certainly finite, if it is convergent in the extreme case, when the modulus ρ^2 is substituted for $\sin^2 \frac{u}{n}$. Then by the foregoing reasoning,

$$R > 1 - n^2\left(\sin^2 \frac{x}{n} + \sinh^2 \frac{y}{n}\right) . \frac{1}{2m} .$$

Now $\sin\frac{x}{n} < \frac{x}{n}$ and $\sinh\frac{y}{n} > \frac{y}{n}$, but the limits of $n\sin\frac{x}{n}$ and of $n\sinh\frac{y}{n}$ are respectively $\frac{\pi}{2}x$ and $\frac{\pi}{2}y$, so that, when n is very large compared with x and y,

$$n^2\left(\sin^2\frac{x}{n} + \sinh^2\frac{y}{n}\right) = x^2 + y^2 + k,$$

where k is a small quantity, which may be either positive or negative depending on the values of x and y, and which vanishes, when n becomes infinite, so that the limiting value of R, when n is infinite, may in this case be written as $1 - \epsilon_1\frac{\pi^2}{8}\frac{x^2+y^2}{m}$, and the infinite product has therefore a finite modulus.

The versor of the product is also definite. For, if

$$\rho'_m i^{v_m} = 1 - \frac{\sin^2\left(\frac{x+yi}{n}\right)}{\sin^2\frac{m}{n}},$$

$$\rho'_m \cos v_m = 1 - \frac{\sin^2\frac{x}{n}\cosh^2\frac{y}{n} - \cos^2\frac{x}{n}\sinh^2\frac{y}{n}}{\sin^2\frac{m}{n}}$$

and

$$\rho'_m \sin v_m = \frac{-2\sin\frac{x}{n}\cos\frac{x}{n}\,.\,\sinh\frac{y}{n}\cosh\frac{y}{n}}{\sin^2\frac{m}{n}}.$$

Whence, after reduction

$$\tan v_m = \frac{\sin\frac{2x}{n}\,.\,\sinh\frac{2y}{n}}{\cos\frac{2m}{n} - \cos\frac{2x}{n}\cosh\frac{2y}{n}}$$

which reduces, when $n = \infty$, to

$$\tan v_m = \frac{-2xy}{m^2 - (x^2 - y^2)}:$$

and since, when m is taken large enough, $\cos v_m$ is positive, the prime value of v_m lies between $-1^{\llcorner}$ and $+1^{\llcorner}$, and therefore $\tan v_m > \frac{\pi}{2}v_m$ numerically, so that the series

$$v_{m+2} + v_{m+4} + v_{m+6} + \dots$$

is numerically less than

$$\frac{2}{\pi}\left\{\frac{2xy}{(m+2)^2-(x^2-y^2)} + \frac{2xy}{(m+4)^2-(x^2-y^2)} + \dots,\right\}$$

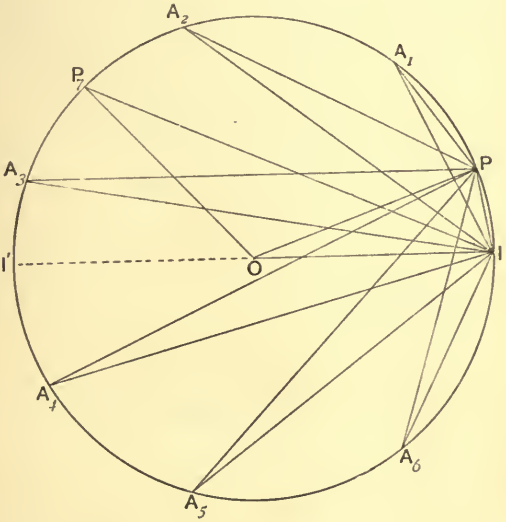
a convergent series. Hence the versor

$$i^{v_1+v_2+v_3+\dots}$$

has a definite value, and therefore, as we have proved the tensor or modulus of the infinite product to be finite, the infinite product has a definite finite limit or is convergent.

6. *Geometrical Illustration.*

The argument of the last article may be well illustrated by a Geometrical Construction.



Let IP_n be a given arc of the unit circle and IP its nth part, (in the figure $n=7$), and let the whole circumference be divided into n equal arcs, IA_1, A_1A_2, $A_2A_3 \dots$: then, as shown in Art. 3,

P and P_n being on the circumference, taking the case of n being an odd number, we have the following relations of the chords,

$$IP_n = IP \,.\, A_1P \,.\, A_2P \,.\,.\,.\, A_{n-1}P$$
$$= IP \times A_1P \,.\, A_{n-1}P \times A_2P \,.\, A_{n-2}P \,.\,.\,. \times A_{\frac{n-1}{2}}P \,.\, A_{\frac{n+1}{2}}P,$$

and $\quad n = A_1I \,.\, A_2I \,.\,.\,.\, A_{n-1}I = A_1I^2 \,.\, A_2I^2 \,.\,.\,.\, A^2_{\frac{n-1}{2}}.$

since from symmetry $A_rI = A_{n-r}I$. Therefore by division

$$\frac{IP_n}{n\,.\,IP} = \frac{A_1P \,.\, A_{n-1}P}{A_1I^2} \,.\, \frac{A_2P \,.\, A_{n-2}P}{A_2I^2} \,.\,.\,. \frac{A_{\frac{n-1}{2}}P \,.\, A_{\frac{n+1}{2}}P}{A^2_{\frac{n-1}{2}}}.$$

Now, as long as m is finite, when n is increased without limit, IP_n being a given finite arc, the chords A_mP, $A_{n-m}P$, A_mI vanish in the ratio of the arcs A_mP, $A_{n-m}P$, A_mI, that is, in the ratio of $A_mI - PI$, $A_mI + PI$, A_mI, so that when m is finite and n infinite

$$\text{lt.}\ \frac{A_mP \,.\, A_{n-m}P}{A_mI^2} = 1 - \frac{(\text{arc}\ IP)^2}{(\text{arc}\ A_mI)^2}.$$

Hence, if arc $IP_n = 2u$ quadrants, so that

$$\text{arc}\ IP = \frac{2u}{n}\ \text{quadrants, and arc}\ A_mI = m \,.\, \frac{4}{n}\ \text{quadrants,}$$

then $\quad IP_n = 2\sin u,\ n\,.\,IP = n\,.\,2\sin\frac{u}{n} = u\pi$ ultimately

and $\quad \text{lt.}\ \dfrac{A_mP \,.\, A_{n-m}P}{A_mI^2} = 1 - \dfrac{\dfrac{4u^2}{n^2}}{\dfrac{16m^2}{n^2}} = 1 - \dfrac{u^2}{4m^2}.$

If however m remains comparable with n, as n is increased, so that m/n is finite, since P ultimately coincides with I, A_mP, $A_{m-n}P$ are both ultimately equal to A_mI, and the limit of $\dfrac{A_mP \,.\, A_{m-n}P}{A_mI^2}$ is 1.

Hence the relation above between the chords drawn from P and I to the A's and the chord IP_n of the arc $2u$ becomes in the limit

$$\frac{\sin u}{u} = \frac{\pi}{2}\left(1-\frac{u^2}{2^2}\right)\left(1-\frac{u^2}{4^2}\right)\left(1-\frac{u^2}{6^2}\right)\ldots \textit{ ad inf.}$$

The corresponding relations for $I'P_n$ and the chords drawn to $B_1, B_2, \ldots$ may be shown in like manner to give the series for $\cos u$.

7. *Factor series for sinh and cosh.*

In the formulæ established in Art. 3 above, put iu for u, then since $\cos iu = \cosh u$, $\sin iu = i \sinh u$, we have

$$\frac{\sinh u}{n \,.\, \sinh \frac{u}{n}} = \left(1+\frac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{2}{n}}\right)\left(1+\frac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{4}{n}}\right)\ldots \begin{cases} \left(1+\dfrac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{n-1}{n}}\right) n \text{ odd}, \\ \left(1+\dfrac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{n-2}{n}}\right) \cosh \dfrac{u}{n}, \\ \qquad\qquad\qquad n \text{ even}, \end{cases}$$

$$\cosh u = \left(1+\frac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{1}{n}}\right)\left(1+\frac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{3}{n}}\right)\ldots \begin{cases} \left(1+\dfrac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{n-2}{n}}\right) \cosh \dfrac{u}{n}, \\ \qquad\qquad\qquad n \text{ odd}, \\ \left(1+\dfrac{\sinh^2 \frac{u}{n}}{\sin^2 \frac{n-1}{n}}\right) n \text{ even}, \end{cases}$$

$$\sinh u = \frac{\pi}{2}\, u \left(1+\frac{u^2}{2^2}\right)\left(1+\frac{u^2}{4^2}\right)\left(1+\frac{u^2}{6^2}\right)\ldots \textit{ ad inf.}$$

$$\cosh u = \left(1 + \frac{u^2}{1^2}\right)\left(1 + \frac{u^2}{3^2}\right)\left(1 + \frac{u^2}{5^2}\right) \ldots \textit{ad inf.}$$

or, using circular measure, since $u = \frac{2}{\pi}\theta$,

$$\sinh \theta = \theta \left(1 + \frac{\theta^2}{\pi^2}\right)\left(1 + \frac{\theta^2}{2^2\pi^2}\right)\left(1 + \frac{\theta^2}{3^2\pi^2}\right) \ldots \textit{ad inf.}$$

$$\cosh \theta = \left(1 + \frac{4\theta^2}{\pi^2}\right)\left(1 + \frac{4\theta^2}{3^2\pi^2}\right)\left(1 + \frac{4\theta^2}{5^2\pi^2}\right) \ldots \textit{ad inf.}$$

These series might, of course, be deduced independently from the formulæ in Art. 2 above, by putting $z = \eta^u$, and $a = \eta^{-u}$, reducing as in the case of the sine and cosine, and with a similar proof of the convergency of the infinite products.

8. *The Factor Series for sine and cosine are periodic.*

The *periodic* character of the series for sine and cosine may be easily shewn as follows.

The formula gives

$$\sin(u+2) = \frac{\pi}{2}(u+2)\left(1 - \frac{(u+2)^2}{2^2}\right)\left(1 - \frac{(u+4)^2}{4^2}\right) \ldots$$

Now

$$1 - \frac{(u+m)^2}{n^2} = \left(1 - \frac{u+m}{n}\right)\left(1 + \frac{u+m}{n}\right)$$

$$= \frac{n-m}{n}\left(1 - \frac{u}{n-m}\right) \cdot \frac{n+m}{n}\left(1 + \frac{u}{n+m}\right),$$

and putting $m = 2$ and $n = 2, 4, 6, \ldots$ successively,

$$\sin(u+2) = \frac{\pi}{2} \cdot 2\left(1 + \frac{u}{2}\right)\left\{\begin{array}{l} \qquad\qquad -\frac{u}{2} \cdot \frac{2}{4}\left(1 - \frac{u}{2}\right) \cdot \frac{4}{6}\left(1 - \frac{u}{4}\right) \ldots \\ \frac{4}{2}\left(1 + \frac{u}{4}\right) \cdot \frac{6}{4}\left(1 + \frac{u}{6}\right) \cdot \frac{8}{6}\left(1 + \frac{u}{8}\right) \ldots \end{array}\right.$$

$$= -\frac{\pi}{2} \cdot u\left(1-\frac{u^2}{2^2}\right)\left(1-\frac{u^2}{4^2}\right)\ldots$$

$$= -\sin u,$$

whence $\sin(u+4) = -\sin(u+2) = \sin u$, &c.

The student will in like manner easily verify the periodicity of the cosine series, the change of the sine series into the cosine series, &c.

9. *Series for $\log_e \sin u$, &c.*

Taking logarithms of both sides of the formula for $\sin u$ in factors,

$$\log_e \sin u = \log_e \frac{\pi}{2} + \log_e u + \log_e\left(1-\frac{u^2}{2^2}\right) + \log_e\left(1-\frac{u^2}{4^2}\right) + \ldots$$

and expanding each term by the form

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3}\ldots,$$

we have

if $\frac{u^2}{2^2}\left(\text{and therefore also } \frac{u^2}{4^2}, \frac{u^2}{6^2}\ldots\right) < 1$, or $u < 2$,

$$\log_e \sin u = \log_e \frac{\pi}{2} + \log_e u - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right)u^2$$

$$-\left(\frac{1}{2^4} + \frac{1}{4^4} + \ldots\right)\frac{u^4}{2} - \left(\frac{1}{2^6} + \frac{1}{4^6} + \ldots\right)\frac{u^6}{3} - \text{\&c.}$$

$$= \log_e \frac{\pi}{2} + \log_e u - S_2 u^2 - \frac{1}{2} S_4 u^4 - \frac{1}{3} S_6 u^6 - \ldots$$

where $S_p = \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \ldots$

Now

$$S_2 = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots < \frac{1}{4} + \frac{1}{2\,.\,4} + \frac{1}{4\,.\,6} + \ldots$$

$$\text{that is, } S_2 < \frac{1}{4} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{6} - \frac{1}{8} + \ldots\right)$$

$$< \frac{1}{4} + \frac{1}{4} \text{ or } \frac{1}{2},$$

$$S_4 < \frac{1}{4} S_2,\ S_6 < \frac{1}{4} S_4,\ S_8 < \frac{1}{4} S_6 \text{ \&c.},$$

and hence the above series is convergent and the formula quantitatively true, if $u < 2$.

Similarly, if $u < 2$,

$$\log_e \sinh u = \log_e \frac{\pi}{2} + \log_e u + S_2 u^2 - \frac{1}{2} S_4 u^4 + \frac{1}{3} S_6 u^6 - \ldots$$

In like manner, if $u < 1$, and $S'_p = \frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \ldots$

$$\log_e \cos u = \log_e \left(1 - \frac{u^2}{1^2}\right) + \log_e \left(1 - \frac{u^2}{3^2}\right) + \log_e \left(1 - \frac{u^2}{5^2}\right) + \ldots$$

$$= -S'_2 u^2 - \frac{1}{2} S'_4 u^4 - \frac{1}{3} S'_6 u^6 - \ldots$$

and $$\log_e \cosh u = S'_2 u^2 - \frac{1}{2} S'_4 u^4 + \frac{1}{3} S'_6 u^6 - \ldots, \text{ if } u \not> 1.$$

By means of these series, the numerical values of log-sin u, log-sinh u, &c. may be found directly, without calculating the values of $\sin u$, $\sinh u$, &c. themselves. The series for log-sine and log-sinh are rapidly convergent, if $u < \frac{1}{2}$, so that they are suitable for the calculation of their values for angles less than 45°, whence the values for angles greater than 45° can be obtained.

Differentiating * the expressions for $\log_e \sin u$ and $\log_e \sinh u$,

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* Or, what is the same thing, taking the limit, when h vanishes, of the equation

$$\frac{\log_e \sin(u+h) - \log_e \sin u}{h} = \frac{\log_e (u+h) - \log_e u}{h}$$
$$- S_2 \frac{(u+h)^2 - u^2}{h} - \frac{1}{2} S_4 \frac{(u+h)^4 - u^4}{h} - \ldots$$

we have

$$\frac{\pi}{2}\cot u = \frac{1}{u} - 2\,(S_2\,u + S_4\,u^3 + S_6\,u^5 + \ldots)$$

$$\frac{\pi}{2}\coth u = \frac{1}{u} + 2\,(S_2\,u - S_4\,u^3 + S_6\,u^5 - \ldots)$$

which satisfy the ordinary test of convergency and therefore are quantitatively valid, if $u^2 < 4$, or u is between -2 and $+2$. If $u = 2$ or -2, the series for $\cot u$ is divergent, and that for $\coth u$ fluctuating.

Also differentiating the expressions for $\log_e \cos u$ and $\log_e \cosh u$, we have

$$\frac{\pi}{2}\tan u = 2\,(S'_2\,u + S'_4\,u^3 + S'_6\,u^5 + \ldots)$$

$$\frac{\pi}{2}\tanh u = 2\,(S'_2\,u - S'_4\,u^3 + S'_6\,u^5 - \ldots)$$

which are quantitatively valid, by the ordinary test if u is between -1 and $+1$, but the series for $\tan u$ divergent and that for $\tanh u$ fluctuating, if $u = 1$ or -1.

From these, putting $u = 1$, we obtain the following :

$$0 = \frac{\pi}{2}\cot 1 = 1 - 2\,(S_2 + S_4 + S_6 + \ldots) \text{ or } S_2 + S_4 + S_6 + \ldots = \frac{1}{2},$$

$$\frac{\pi}{2}\coth 1 = \frac{\pi}{2}\cdot\frac{\eta^2+1}{\eta^2-1} = 1 + 2\,(S_2 - S_4 + S_6 - \ldots)$$
$$= 4\,(S_2 + S_6 + S_{10} + \ldots),$$

whence the value of η might be directly calculated, the values of S_2, S_6, . . . being known.

Also

$$\frac{\pi}{2}\tanh 1 = \frac{\pi}{2}\cdot\frac{\eta^2-1}{\eta^2+1} = 2\,(S'_2 - S'_4 + S'_6 - S'_8 + \ldots).$$

It is to be borne in mind that differentiation, or this process of taking limits, *may* in certain cases convert a convergent series into a fluctuant or divergent series, or alter the limits of convergency. This cannot be the case however in convergent series, proceeding by integral powers of x (scalar), differentiated with respect to x, except *at* the limit of convergency.

Again since

$$\frac{\pi}{4}\frac{\tan u}{u} = S'_2 + S'_4\, u^2 + S'_6\, u^4 + \ldots, \text{ and } \mathrm{lt.}_{u=0}\left(\frac{\tan u}{u}\right) = \frac{\pi}{2},$$

putting $u = 0$, $\dfrac{\pi^2}{8} = S'_2$.

10. *Wallis's Theorem.*

In the series for $\dfrac{\sin u}{u}$, putting $u = 1$,

$$1 = \frac{\pi}{2} \,.\, 1 \,.\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\ldots$$

whence

$$\frac{\pi}{2} = \frac{2^2}{1\,.\,3} \cdot \frac{4^2}{3\,.\,5} \cdot \frac{6^2}{5\,.\,7} \cdots \frac{(2p)^2}{(2p-1)\,(2p+1)}$$

ultimately, when $p = \infty$ a result which is known as Wallis's Theorem.

Hence, when p is a large number,

$$\frac{2\,.\,4\,.\,6\,\ldots\,2p}{1\,.\,3\,.\,5\,\ldots\,(2p-1)} = \sqrt{\pi\left(p + \frac{1}{2}\right)} \text{ or } \sqrt{\pi p} \text{ nearly,}$$

since $\frac{1}{2}$ may be neglected in comparison with p, when p is large.

[The following deduction of an important theorem, first given by Stirling, for an approximation to the value of $\underline{|n}$, when n is a large number, is taken from De Morgan. It does not profess to be a valid proof, but it is one which may give the student some confidence in the formula, pending the more difficult and complete proofs given in works on higher analysis. The theorem is that, when n is a large number, $1\,.\,2\,.\,3\ldots n$ or $\underline{|n} = \sqrt{2\pi n}\,.\left(\dfrac{n}{e}\right)^n$ approximately.

Let $\underline{|n} = n^n \,.\, \phi n$ or

$$\phi n = 1\,.\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right),$$

a small quantity when n is large, and evanescent, when n is infinite.

Then

$$\frac{2 \, . \, 4 \, . \, 6 \ldots 2n}{1 \, . \, 3 \, . \, 5 \ldots 2n-1} = \frac{(2 \, . \, 4 \, . \, 6 \ldots 2n)^2}{1 \, . \, 2 \, . \, 3 \ldots (2n-1) 2n} = \frac{2^{2n} \, . \, (\underline{|n})^2}{\underline{|2n}}$$

$$= \frac{2^{2n} \, . \, n^{2n} (\phi n)^2}{(2n)^{2n} \phi (2n)} = \frac{(\phi n)^2}{\phi (2n)}$$

so that, by Wallis's theorem $(\phi n)^2 = \sqrt{\pi n} \, . \, \phi (2n)$ nearly, when n is large, or dividing by $2\pi n$,

$$\left(\frac{\phi n}{\sqrt{2\pi n}} \right)^2 = \frac{\phi (2n)}{\sqrt{2\pi \, . \, 2n}}$$

so that, if $fn = \dfrac{\phi n}{\sqrt{2\pi n}}$, and n is large, $\underline{|n} = \sqrt{2\pi n} \, . \, n^n \, . \, fn$

nearly, where $(fn)^2 = f(2n)$, n being a positive integer.

Assuming that no important error would be introduced by treating this approximate relation as accurately true, it is easy to find the form of fn. Taking logarithms, we have

$$2 \log (fn) = \log f(2n),$$

and dividing by $2n$,

$$\frac{\log (fn)}{n} = \frac{\log f(2n)}{2n},$$

so that $\dfrac{\log (fn)}{n}$ is either a constant independent of n, or a function of n which (like $\cos 4 \log n / \log 2$) is unchanged, when $2n$ is put for n. *Assuming* it to be constant at and about a given large value of n, we have, if $\dfrac{\log fn}{n} = \log k$, $fn = k^n$, so that $\underline{|n} = \sqrt{2\pi n} \, . \, n^n \, . \, k^n$, where k is a constant to be determined. From this

$$\underline{|n+1} = \sqrt{2\pi (n+1)} \, . \, (n+1)^{n+1} \, . \, k^{n+1},$$

$$\text{whence } n+1 = \frac{\underline{|n+1}}{\underline{|n}} = \sqrt{\frac{n+1}{n}} \, . \left(\frac{n+1}{n} \right)^n . \, (n+1) \, . \, k$$

$$\text{or } 1 = \left(1 + \frac{1}{n} \right)^{\frac{1}{2}} \left(1 + \frac{1}{n} \right)^n . \, k,$$

but, when n is large, $\left(1 + \dfrac{1}{n} \right)^{\frac{1}{2}} = 1$ nearly and $\left(1 + \dfrac{1}{n} \right)^n = e$ nearly so that approximately $$k=e^{-1}$$

and $$\underline{|n} = \sqrt{2\pi n}\, n^n . e^{-n}.$$

A closer approximation, obtained by accurate reasoning, is

$$\underline{|n} = \sqrt{2\pi n} . n^n c^{-n+\frac{1}{12n}+\theta}$$

where θ is a small quantity of the order $\left.\frac{1}{n^2}\right]$.

11. *Other deductions from the series for sine, &c.*

We have

$$\frac{\sin u}{u} = \frac{\pi}{2} . \left(1-\frac{u^2}{2^2}\right)\left(1-\frac{u^2}{4^2}\right)\left(1-\frac{u^2}{6^2}\right)+ . . .$$

$$={}^{\pi}\ (1 - S_2\, u^2 + \text{terms in } u^4, u^6 . . .)$$

Also

$$\frac{\sin u}{u} = \frac{\pi}{2}\left\{1-\frac{\left(\frac{\pi}{2}u\right)^2}{\underline{|3}}+\frac{\left(\frac{\pi}{2}u\right)^4}{\underline{|5}} - . . .\right\}$$

hence, equating the coefficients of u^2 in these two series,

$$S_2 = \frac{1}{\underline{|3}}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{24} \text{ or } \frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+ . . . \textit{ ad inf.} = \frac{\pi^2}{24}.$$

Multiplying by 2^2,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ . . . =\frac{\pi^2}{6}$$

whence

$$S'_2 = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ . . . = \frac{\pi^2}{6}-\frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

The same results might be in like manner obtained from the series for $\cos u$ or from those for $\cosh u$ or $\sinh u$.

12. *Relations between Bernoulli's numbers and sums of inverse powers of the natural numbers.*

Comparing the formulæ obtained in Art. 9 for $\cot u$ (after changing to circular measure, or putting $\frac{2x}{\pi}$ for u) with that in Chap. IX. Art. 18, namely :

$$\cot x = \frac{1}{x} - \frac{2^3}{\pi^2} S_2 x - \frac{2^5}{\pi^4} S_4 x^3 - \frac{2^7}{\pi^6} S_6 x^5 - .$$

with

$$\cot x = \frac{1}{x} - \frac{B_2}{\underline{|2}} . 2^2 x - \frac{B_4}{\underline{|4}} 2^4 x^3 - \frac{B_6}{\underline{|6}} . 2^6 x^5 - \ldots$$

we find, by equating coefficients of like powers,

$$\frac{2S_2}{\pi^2} = \frac{B_2}{\underline{|2}}, \quad \frac{2S_4}{\pi^4} = \frac{B_4}{\underline{|4}}, \ldots \quad \frac{2S_{2r}}{\pi^{2r}} = \frac{B_{2r}}{\underline{|2r}} \ldots$$

whence, using the values of B_2, B_4 . . . as found in Chap. IX. § 17,

$$S_2 = \frac{\pi^2}{24}, \; S_4 = \frac{\pi^4}{1440}, \; S_6 = \frac{\pi^6}{60480}, \; S_8 = \frac{\pi^8}{2419200}, \; S_{10} = \frac{\pi^{10}}{95800320} \ldots$$

or

$$S_2 = \cdot 4112336, \; S_4 = \cdot 0676452, \; S_6 = \cdot 0158960$$
$$S_8 = \cdot 0039222, \; S_{10} = \cdot 00097753 \ldots$$

Hence, if $$\Sigma_r = \frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \ldots = 2^r S_r$$

$$\Sigma_2 = 1 \cdot 644934, \; \Sigma_4 = 1 \cdot 082323, \; \Sigma_6 = 1 \cdot 018944,$$
$$\Sigma_8 = 1 \cdot 004084, \; \Sigma_{10} = 1 \cdot 0009904$$

and, if $$S'_r = \frac{1}{1^r} + \frac{1}{3^r} + \frac{1}{5^r} + \ldots$$

$$S'_2 = 1 \cdot 233700, \; S'_4 = 1 \cdot 014678, \; S'_6 = 1 \cdot 003048$$
$$S'_8 = 1 \cdot 000162, \; S'_{10} = 1 \cdot 0000129.$$

13. *Limits of Convergency of the series dependent on Bernoulli's numbers.*

Since

$$S_{2r} = \frac{1}{2^{2r}} \Sigma_{2r} \text{ and } S_{2r+2} = \frac{1}{2^{2r+2}} \Sigma_{2r+2}$$

$$\frac{S_{2r+2}}{S_{2r}} = \tfrac{1}{4} \cdot \frac{\Sigma_{2r+2}}{\Sigma_{2r}}$$

but $\Sigma_{2r+2} / \Sigma_{2r} < 1$ for any finite value of r, and ultimately $= 1$, when $r = \infty$, therefore $S_{2r+2} / S_{2r} < \frac{1}{4}$ for any finite value of r, and ultimately $= \frac{1}{4}$, when $r = \infty$. Also

$$B_{2r} = 2S_{2r} \cdot \frac{\underline{|2r}}{\pi^{2r}}, \text{ whence } \frac{B_{2r+2}}{B_{2r}} = \frac{S_{2r+2}}{S_{2r}} \cdot \frac{(2r+1)(2r+2)}{\pi^2}$$

so that

$$\frac{B_{2r+2}}{B_{2r}} < \frac{(2r+1)(2r+2)}{(2\pi)^2}$$

if r is finite and $= \dfrac{r^2}{\pi^2}$ nearly, if r is large.

Hence Bernoulli's numbers form a series of decreasing fractions as far as B_6 inclusive, after which they increase continually without limit.

In the expansion of $\dfrac{1}{e^x - 1}$ (Chap. IX. § 18), the coefficient of the term involving x^{2r-1} is $\dfrac{B_{2r}}{\underline{|2r}}$, and the ratio of the next following term to this $= \dfrac{B_{2r+2}}{B_{2r}} \cdot \dfrac{\underline{|2r}}{\underline{|2r+2}} x^2$ and is therefore less than $\dfrac{x^2}{4\pi^2}$, if r is finite, and ultimately $= \dfrac{x^2}{4\pi^2}$, if $r = \infty$. Hence the series is absolutely convergent if x lies between -2π and $+2\pi$.

In the series for $\coth x$ and $\cot x$, the ratio of the term convolving x^{2r+1} to the preceding term

$$=\frac{B_{2r+2}}{B_{2r}}\cdot\frac{\left|\underline{2r}\right.}{\left|\underline{2r+2}\right.}\cdot\frac{2^{2r+2}}{2^{2r}}\,x^2=\frac{x^2}{\pi^2} \text{ ultimately, when } r=\infty,$$

so that these series are both convergent for values of x between $-\pi$ and $+\pi$. The same is seen in like manner to be true for the series for $\operatorname{cosech} x$ and $\operatorname{cosec} x$. Those for $\tanh x$ and $\tan x$ however are convergent only between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

EXAMPLES.

Prove the following formulæ :—

(1) $$\cos\frac{\theta}{2}\,.\,\cos\frac{\theta}{2^2}\,.\,\cos\frac{\theta}{2^3}\ \ldots\ \textit{ad inf.} = \frac{\sin\theta}{\theta}.$$

(2) $$\left(1 - \tan^2\frac{\theta}{2}\right)\left(1 - \tan^2\frac{\theta}{2^2}\right)\left(1 - \tan^2\frac{\theta}{2^3}\right)\ \ldots\ \textit{ad inf.} = \frac{\theta}{\tan\theta}$$

and

$$\left(1 + \tanh^2\frac{\theta}{2}\right)\left(1 + \tanh^2\frac{\theta}{2^2}\right)\ \ldots\ \textit{ad inf.} = \frac{\theta}{\tanh\theta}.$$

(3) $$\sin\frac{2}{n}\,.\,\sin\frac{4}{n}\,.\,\sin\frac{6}{n}\ \ldots \begin{cases}\sin\dfrac{n-1}{n}\ (n\text{ odd})\\[2ex] \sin\dfrac{n-2}{n}\ (n\text{ even})\end{cases} = \frac{\sqrt{n}}{2^{\frac{n-1}{2}}}$$

and

$$\text{log-sin } 10^0 + \text{log-sin } 20^0 + \text{log-sin } 30^0 + \ldots$$
$$+ \text{log-sin } 80^0 = \log 3 - 8\log 2.$$

(4) If n is an odd number,

$$\text{log-cos}\frac{2}{n} + \text{log-cos}\frac{4}{n} + \ldots + \text{log-cos}\frac{n-1}{n} = -\frac{n-1}{2}\,.\,\log 2.$$

(5) $$\sin nu = 2^{n-1}\sin u\,.\,\sin\left(u + \frac{2}{n}\right).\sin\left(u + \frac{4}{n}\right)$$
$$\ldots\sin\left(u + \frac{2(n-1)}{n}\right),$$

and

$$\tan u\,.\,\tan\left(u + \frac{2}{n}\right).\tan\left(u + \frac{4}{n}\right)\ldots\tan\left(u + \frac{2(n-1)}{n}\right)$$
$$= (-1)^{\frac{n-1}{2}}\,.\,\tan nu, \text{ if } n \text{ is odd.}$$

(6) $1 - \dfrac{\sin^2\alpha}{\sin^2\beta} = \cos^2\alpha\left(1 - \dfrac{\tan^2\alpha}{\tan^2\beta}\right)$, and hence show that ($n$ being odd),

$$\sin u = \left(\cos\frac{u}{n}\right)^n.\,n\tan\frac{u}{n}\left(1 - \frac{\tan^2\frac{u}{2}}{\tan^2\frac{2}{n}}\right)\left(1 - \frac{\tan^2\frac{u}{n}}{\tan^2\frac{4}{n}}\right)\ldots$$
$$\left(1 - \frac{\tan^2\frac{u}{n}}{\tan^2\frac{n-1}{n}}\right).$$

(7) From Ex. 6, show that, if $u \not> 2$,

$$\text{and } P_n = \frac{\pi}{2} u \left(1 - \frac{u^2}{2^2}\right) \dots \left(1 - \frac{u^2}{(n-1)^2}\right),$$

$$\sin u < P_n \text{ and } > P_n \left(\cos \frac{u}{n}\right)^n\text{: whence } \sin u = P_\infty .$$

(8) $\sin u = u \,.\, \Pi_1^\infty \left\{ \frac{(2n-u)(2n+u)}{(2n-1)(2n+1)} \right\}$

$$\text{and } \tan u = \frac{u}{1-u} \Pi_1 \; \frac{(2n-u)(2n+u)}{(2n-1+u)(2n+1-u)} .$$

(9) $(1 + \sec u)\left(1 + \sec \frac{u}{2}\right)\left(1 + \sec \frac{u}{2^2}\right) \dots \left(1 + \sec \frac{u}{2^n}\right)$

$$= \tan u \,.\, \cot \frac{u}{2^{n+1}},$$

whence $$\frac{\pi}{4} = \frac{2}{1 + \sec \frac{1}{2}} \cdot \frac{2}{1 + \sec \frac{1}{4}} \cdot \frac{2}{1 + \sec \frac{1}{8}} \dots \textit{ad inf.}$$

and $$\pi = 2 \sec \frac{1}{2} \,.\, \sec \frac{1}{4} \,.\, \sec \frac{1}{8} \,.\, \sec \frac{1}{16} \dots \textit{ad inf.}$$

(10) $\left(1 + 4 \sinh^2 \frac{u}{3}\right)\left(1 + 4 \sinh^2 \frac{u}{3^2}\right)\left(1 + 4 \sinh^2 \frac{u}{3^3}\right) \dots \textit{ad inf.} = \cosh \theta.$

(11) From the formulæ of § 4, prove that

$$\cos v + \cos u = 2 \cos^2 \frac{u}{2} \,.\, \left(1 - \frac{\sin^2 \frac{v}{2n}}{\sin^2 \frac{2+u}{2n}}\right)\left(1 - \frac{\sin^2 \frac{v}{2n}}{\sin^2 \frac{6+u}{2n}}\right)$$

$$\dots \left(1 - \frac{\sin^2 \frac{v}{2n}}{\sin^2 \frac{2(2n-1)+u}{2n}}\right).$$

Why cannot it be inferred from this, by taking $n = \infty$, that

$$\cos v + \cos u = 2 \cos^2 \frac{u}{2} \,.\, \left(1 - \frac{v^2}{(2+u)^2}\right) . \left(1 - \frac{v^2}{(6+u)^2}\right) .$$

$$\left(1 - \frac{v^2}{(10+u)^2}\right) \dots \textit{ad inf.}\,?$$

Prove that the true result is

$$\cos v + \cos u = 2\cos^2\frac{u}{2} \cdot \left(1 - \frac{v^2}{(2+u)^2}\right) \cdot \left(1 - \frac{v^2}{(2-u)^2}\right) \cdot$$
$$\left(1 - \frac{v^2}{(6+u)^2}\right) \cdot \left(1 - \frac{v^2}{(6-u)^2}\right) \ldots \textit{ ad inf.}$$

(12) $$\frac{\sinh(u+v)}{\sinh u} = \left(1 + \frac{v}{u}\right)\left(1 + \frac{2uv+v^2}{2^2+u^2}\right)\left(1 + \frac{2uv+v^2}{4^2+u^2}\right)$$
$$\left(1 + \frac{2uv+v^2}{6^2+u^2}\right) \ldots$$

(13) $$\frac{\sinh v}{v} = \frac{\sinh u}{u} \cdot \left(1 + \frac{v^2-u^2}{2^2+u^2}\right)\left(1 + \frac{v^2-u^2}{4^2+u^2}\right)\left(1 + \frac{v^2-u^2}{6^2+u^2}\right) \ldots$$

(14) $$\frac{\sinh u}{\sin u} = \left(1 + \frac{2u^2}{2^2-u^2}\right)\left(1 + \frac{2u^2}{4^2-u^2}\right)\left(1 + \frac{2u^2}{6^2-u^2}\right) \ldots$$

(15) $$\epsilon^{\frac{\pi}{2}} \text{ or } \eta = \left\{\left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{5^2}\right) \ldots\right\}$$
$$+ \left\{\left(1 + \frac{2}{1\,.\,3}\right)\left(1 + \frac{2}{3\,.\,5}\right)\left(1 + \frac{2}{5\,.\,7}\right) \ldots\right\}.$$

(16) $$\cosh v + \cosh u = 2\cosh^2\frac{u}{2} \cdot \left(1 + \frac{v^2+2uv}{2^2+u^2}\right)\left(1 + \frac{v^2-2uv}{2^2+u^2}\right)$$
$$\left(1 + \frac{v^2+2uv}{6^2+u^2}\right)\left(1 + \frac{v^2-2uv}{6^2+u^2}\right) \ldots$$

(17) $$\cos u + \sin u = \left(1 + \frac{4u(1+u)}{1\,.\,3}\right)\left(1 + \frac{4u(1-u)}{5\,.\,7}\right)$$
$$\left(1 + \frac{4u(1-u)}{9\,.\,11}\right) \ldots$$

and

$$\cos u - \sin u = \left(1 - \frac{4u(1+u)}{1\,.\,3}\right)\left(1 - \frac{4u(1+u)}{5\,.\,7}\right)\left(1 - \frac{4u(1+u)}{9\,.\,11}\right) \ldots$$

Hence verify the formula, $\cos 2u = \cos^2 u - \sin^2 u$.

(18) By the factor series for sine and cosine, verify the formula

$$\sin 2u = 2\sin u \cos u.$$

(19) Prove that, if n is a *prime* number,

$$\Pi_2^\infty \frac{n^2}{n^2-1} \text{ or } \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdots$$

$$= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6} \text{ (Crofton),}$$

or

$$\pi^2 = 6\times\left(1+\frac{1}{3}\right)\left(1+\frac{1}{8}\right)\left(1+\frac{1}{24}\right)\ldots\left(1+\frac{1}{n^2-1}\right)\ldots$$

(20) $$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots = \frac{\pi^4}{90} = \left(1+\frac{1}{2^4-1}\right)\left(1+\frac{1}{3^4-1}\right)\ldots$$

$$\left(1+\frac{1}{n^4-1}\right)\ldots, \ n \text{ a } \textit{prime} \text{ number.}$$

(21) $$\frac{\pi^2}{15} = \left(1-\frac{1}{2^2+1}\right)\left(1-\frac{1}{3^2+1}\right)\left(1-\frac{1}{5^2+1}\right)\ldots$$

$$\left(1-\frac{1}{n^2+1}\right)\ldots, \ n \text{ a prime number.}$$

(22) $$\left(1+\frac{1}{2^2-1}\right)\left(1+\frac{1}{6^2-1}\right)\left(1+\frac{1}{10^2-1}\right)\ldots = \sqrt{2}$$

(23) $$\pi = 3\left(1+\frac{1}{6^2-1}\right)\left(1+\frac{1}{12^2-1}\right)\left(1+\frac{1}{18^2-1}\right)$$

$$\left(1+\frac{1}{24^2-1}\right)\ldots$$

(24) $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots = \frac{\pi^2}{12}.$$

(25) $$\frac{B_{2r}}{\underline{|2r}} = \frac{2}{(2\pi)^{2r}}\left(1+\frac{1}{2^{2r}-1}\right)\left(1+\frac{1}{3^{2r}-1}\right)\left(1+\frac{1}{5^{2r}-1}\right)\ldots$$

$$\left(1+\frac{1}{n^{2r}-1}\right)\ldots, \ n \text{ a prime number.}$$

CHAPTER XI

SERIES OF PARTIAL FRACTIONS

1. *Resolution of* $1/z^n-1$ *into the sum of* n *fractions.*

We have seen (Chap. IX. § 2) that z^n-1 can be resolved into n factors of the form $z-i^{\frac{4\lambda}{n}}$ for n consecutive integral values of λ: hence its reciprocal may be expressed as the sum of n fractions of the form $\dfrac{A_\lambda}{z-\alpha^\lambda}$, if $\alpha=i^{\frac{4}{n}}\left(\text{the } \textit{prime} \text{ value of } 1^{\frac{1}{n}}\right)$, where A_λ is a quantity (independent of z) to be determined.

We have then

$$\frac{1}{z^n-1}=\frac{A_0}{z-1}+\frac{A_1}{z-\alpha}+\ldots+\frac{A_\lambda}{z-\alpha^\lambda}+\ldots+\frac{A_{n-1}}{z-\alpha^{n-1}}.$$

Multiplying by $z-\alpha^\lambda$, and putting $z=\alpha^\lambda$, we obtain

$$A_\lambda=\text{limit}\,\frac{z-\alpha^\lambda}{z^n-1},\text{ when } z=\alpha_\lambda,$$

but

$$\frac{z-\alpha^\lambda}{z^n-1}=\frac{z-\alpha^\lambda}{z^n-\alpha^{n\lambda}}=\frac{1}{z^{n-1}+z^{n-2}\,\alpha^\lambda+z^{n-3}\,\alpha^{2\lambda}+\ldots\;+\alpha^{(n-1)\lambda}}$$

therefore $$A_\lambda=\frac{1}{n\alpha^{(n-1)\lambda}}=\frac{\alpha^\lambda}{n\alpha^{n\lambda}}=\frac{\alpha^\lambda}{n},$$

since $$\alpha^{n\lambda}=\left(i^{\frac{4}{n}}\right)^{n\lambda}=i^{4\lambda}=1,$$

so that $\dfrac{n}{z^n-1}=\dfrac{1}{z-1}+\dfrac{a}{z-a}+\dfrac{a^2}{z-a^2}+\dots+\dfrac{a^{n-1}}{z-a^{n-1}}$.

Another form may be given to this result by grouping the terms in pairs as follows :—

Since $a^n=1$, $\dfrac{a^{n-p}}{z-a^{n-p}}=\dfrac{a^{-p}}{z-a^{-p}}$, and we may write

$$\frac{n}{z^n-1}=\frac{1}{z-1}+\left\{\begin{array}{l}\dfrac{a}{z-a}+\dfrac{a^2}{z-a^2}+\dots+\dfrac{a^{\frac{n-1}{2}}}{z-a^{\frac{n-1}{2}}}\\[3ex] \dfrac{a^{-1}}{z-a^{-1}}+\dfrac{a^{-2}}{z-a^{-2}}+\dots+\dfrac{a^{-\frac{n-1}{2}}}{z-a^{-\frac{n-1}{2}}}\end{array}\right\}(n\text{ odd}).$$

$$=\frac{1}{z-1}+\left\{\begin{array}{l}\dfrac{a}{z-a}+\dfrac{a^2}{z-a^2}+\dots+\dfrac{a^{\frac{n-2}{2}}}{z-a^{\frac{n-2}{2}}}\\[3ex] \dfrac{a^{-1}}{z-a^{-1}}+\dfrac{a^{-2}}{z-a^{-2}}+\dots+\dfrac{a^{-\frac{n-2}{2}}}{z-a^{-\frac{n-2}{2}}}\end{array}\right\}-\frac{1}{z+1}\quad(n\text{ even}).$$

Farther, since

$$\frac{a}{z-a}+\frac{a^{-1}}{z-a^{-1}}=\frac{(a+a^{-1})z-2}{z^2-(a+a^{-1})z+1}=\frac{2\left(z\cos\frac{4}{n}-1\right)}{z^2-2z\cos\frac{4}{n}+1},\ \&\text{c.},$$

$$\frac{n}{z^n-1}=\frac{1}{z-1}+\frac{2\left(z\cos\frac{4}{n}-1\right)}{z^2-2z\cos\frac{4}{n}+1}+\frac{2\left(z\cos\frac{8}{n}-1\right)}{z^2-2z\cos\frac{8}{n}+1}+\dots$$

$$+\frac{2\left(z\cos\frac{2(n-1)}{n}-1\right)}{z^2-2z\cos\frac{2(n-1)}{n}+1}\ (n\text{ odd}).$$

$$\frac{n}{z^n-1}=\frac{2}{z^2-1}+\frac{2\left(z\cos\frac{4}{n}-1\right)}{z^2-2z\cos\frac{4}{n}+1}+\frac{2\left(z\cos\frac{8}{n}-1\right)}{z^2-2z\cos\frac{8}{n}+1}+\dots$$

$$+\frac{2\left(z\cos\frac{2\,n-2}{n}-1\right)}{z^2-2z\cos\frac{2\,n-2}{n}+1}\ (n\text{ even}).$$

which are scalar formulæ,* if z is scalar.

2. *Resolution of* $1/z^n+1$ *into the sum of* n *fractions.*

The corresponding formulæ for $1/z^n+1$ (the details of the proof, or the deduction of the formulæ from the last Article, being left to the student) are, if $\alpha=i^{2/n}$ so that $\alpha^n=-1$,

$$\frac{n}{z^n+1}=-\frac{\alpha}{z-\alpha}-\frac{\alpha^3}{z-\alpha^3}-\dots-\frac{\alpha^{2n-1}}{z-\alpha^{2n-1}},$$

$$=\left.\begin{array}{l}-\dfrac{\alpha}{z-\alpha}-\dfrac{\alpha^3}{z-\alpha^3}-\dots-\dfrac{\alpha^{n-2}}{z-\alpha^{n-2}}\\[2ex]-\dfrac{\alpha^{-1}}{z-\alpha^{-1}}-\dfrac{\alpha^{-3}}{z-\alpha^{-3}}-\dots-\dfrac{\alpha^{-(n-2)}}{z-\alpha^{-(n-2)}}\end{array}\right\}+\frac{1}{z+1}\ (n\text{ odd}).$$

$$=\left.\begin{array}{l}-\dfrac{\alpha}{z-\alpha}-\dfrac{\alpha^3}{z-\alpha^3}-\dots-\dfrac{\alpha^{n-1}}{z-\alpha^{n-1}}\\[2ex]-\dfrac{\alpha^{-1}}{z-\alpha^{-1}}-\dfrac{\alpha^{-3}}{z-\alpha^{-3}}-\dots-\dfrac{\alpha^{-(n-1)}}{z-\alpha^{-(n-1)}}\end{array}\right\}\ (n\text{ even}).$$

---

* These results, as most of the formulæ of this chapter, may readily be obtained from the factor formulæ of the preceding chapter by a slight transformation after taking logarithms and differentiating.

$$\frac{n}{z^n+1} = \frac{2\left(1 - z\cos\frac{2}{n}\right)}{1 - 2z\cos\frac{2}{n} + z^2} + \frac{2\left(1 - z\cos\frac{6}{n}\right)}{1 - 2z\cos\frac{6}{n} + z^2} + \ldots$$

$$+ \frac{2\left(1 - z\cos\frac{2(n-2)}{n}\right)}{1 - 2z\cos\frac{2(n-2)}{n} + z^2} + \frac{1}{z+1}\ (n \text{ odd}).$$

$$= \frac{2\left(1 - z\cos\frac{2}{n}\right)}{1 - 2z\cos\frac{2}{n} + z^2} + \frac{2\left(1 - z\cos\frac{6}{n}\right)}{1 - 2z\cos\frac{6}{n} + z^2} + \ldots$$

$$+ \frac{2\left(1 - z\cos\frac{2(n-1)}{n}\right)}{1 - 2z\cos\frac{2(n-1)}{n} + z^2}(n \text{ even}).$$

3. *Cot u and tan u expressed as the mean of* n *cotangents.*

In the series obtained in § 1, put $z = i^{2\frac{u}{n}}$; then

$$\frac{1}{z^n - 1} = \frac{1}{i^{2u} - 1} = \frac{i^{-u}}{i^u - i^{-u}} = \frac{i^{-(1+u)}}{2\sin u}$$

$$= \frac{\cos(1+u) - i\sin(1+u)}{2\sin u} = -\frac{1}{2} - i\,.\,\frac{1}{2}\cot u.$$

$$\frac{a}{z-a} = \frac{1}{\frac{z}{a} - 1} = \frac{1}{i^{\frac{2u-4}{n}} - 1} = \frac{i^{-\frac{u-2}{n}}}{i^{\frac{u-2}{n}} - i^{-\frac{u-2}{n}}} = -\frac{1}{2} - i\,.\,\frac{1}{2}\cot\left(\frac{u-2}{n}\right)$$

$$\frac{a^{-1}}{z - a^{-1}} = \frac{1}{za - 1} = \frac{1}{i^{\frac{2u+4}{n}} - 1} = \frac{i^{-\frac{u+2}{n}}}{i^{\frac{u+2}{n}} - i^{-\frac{u+2}{n}}} = -\frac{1}{2} - i\,.\,\frac{1}{2}\cot\left(\frac{u+2}{n}\right)$$

with similar reductions for the other terms; and it is evident that the sum of the projects on both sides of the equation $= -\frac{n}{2}$, and equating the trajects we have

$$n \cot u = \left.\begin{array}{l} \cot \frac{u}{u} - \cot \left(\frac{2-u}{n}\right) - \cot \frac{4-u}{n} \ldots \\ \qquad\qquad\qquad - \cot \frac{n-1-u}{2} \\ \qquad + \cot \frac{2+u}{n} + \cot \frac{4+u}{n} \ldots \\ \qquad\qquad\qquad + \cot \frac{n-1+u}{n} \end{array}\right\} (n \text{ odd}).$$

$$= \left.\begin{array}{l} \cot \frac{u}{n} - \cot \frac{2-u}{n} - \cot \frac{4-u}{n} \ldots \\ \qquad\qquad\qquad - \cot \left(\frac{n-2-u}{n}\right) \\ \qquad + \cot \frac{2+u}{n} + \cot \frac{4+u}{n} \ldots \\ \qquad\qquad\qquad + \cot \frac{n-2+u}{n} \end{array}\right\} \begin{array}{l} - \tan \frac{u}{n} \\ (n \text{ even}). \end{array}$$

A similar reduction from the series in § 2, or the substitution of $1-u$ for u in the result just obtained, gives

$$n \tan u = \left.\begin{array}{l} \cot \frac{1-u}{n} + \cot \frac{3-u}{n} + \ldots + \cot \frac{n-2-u}{n} \\ - \cot \frac{1+u}{n} - \cot \frac{3+u}{n} + \ldots - \cot \frac{n-2+u}{n} \end{array}\right\} \begin{array}{l} + \tan \frac{u}{n} \\ (n \text{ odd}). \end{array}$$

$$= \left.\begin{array}{l} \cot \frac{1-u}{n} + \cot \frac{3-u}{n} + \ldots + \cot \frac{n-1-u}{n} \\ - \cot \frac{1+u}{n} - \cot \frac{3+u}{n} - \ldots - \cot \frac{n-1+u}{n} \end{array}\right\} (n \text{ even}).$$

Also since

$$\cot(u-v)-\cot(u+v)=\frac{\sin(\overline{u+v}-\overline{u-v})}{\sin(u+v)\sin(u-v)}=\frac{\sin 2v}{\sin^2 u-\sin^2 v},$$

the series for $\cot u$, $\tan u$ may be reduced to

$$\cot u=\frac{1}{n}\cot\frac{u}{n}-\frac{1}{n}\frac{\sin\frac{2u}{n}}{\sin^2\frac{2}{n}-\sin^2\frac{u}{n}}-\frac{1}{n}\cdot\frac{\sin\frac{2u}{n}}{\sin^2\frac{4}{n}-\sin^2\frac{u}{n}}\cdots$$

$$-\frac{1}{n}\frac{\sin\frac{2u}{n}}{\sin^2\frac{n-1}{n}-\sin^2\frac{u}{n}}\ (n\text{ odd}).$$

$$=\frac{1}{n}\cot\frac{u}{n}-\frac{1}{n}\frac{\sin\frac{2u}{n}}{\sin^2\frac{2}{n}-\sin^2\frac{u}{n}}-\frac{1}{n}\cdot\frac{\sin\frac{2u}{n}}{\sin^2\frac{4}{n}-\sin^2\frac{u}{n}}-\cdots$$

$$-\frac{1}{n}\frac{\sin\frac{2u}{n}}{\sin^2\frac{n-2}{n}-\sin^2\frac{u}{n}}-\frac{1}{n}\tan\frac{u}{n}\ (n\text{ even}).$$

$$\tan u=\frac{1}{n}\frac{\sin\frac{2u}{n}}{\sin^2\frac{1}{n}-\sin^2\frac{u}{n}}+\frac{1}{n}\frac{\sin\frac{2u}{n}}{\sin^2\frac{3}{n}-\sin^2\frac{u}{n}}+\cdots$$

$$+\frac{1}{n}\cdot\frac{\sin\frac{2u}{n}}{\sin^2\frac{n-2}{n}-\sin^2\frac{u}{n}}+\frac{1}{n}\tan\frac{u}{n}\ (n\text{ odd}).$$

$$\tan u = \frac{1}{n}\,\frac{\sin\frac{2u}{n}}{\sin^2\frac{1}{n} - \sin^2\frac{u}{n}} + \frac{1}{n}\cdot\frac{\sin\frac{2u}{n}}{\sin^2\frac{3}{n} - \sin^2\frac{u}{n}} + \dots$$

$$+ \frac{1}{n}\,\frac{\sin\frac{2u}{n}}{\sin^2\frac{n-1}{n} - \sin^2\frac{u}{n}} \ (n \text{ even}).$$

4. *Geometrical Interpretation.*

The reasoning of the preceding article is well illustrated by a geometrical interpretation.

Let $\overline{OZ}$, a radius of the unit circle, $= z^n$ or i^{2u} so that $IOZ = 2u^{\llcorner}$, then the chord $\overline{IZ} = z^n - 1$: so also if $\overline{OZ_0} = z$ or $i^{\frac{2u}{n}}$, $\overline{IZ_0} = z - 1$. Then if the arcs $Z_0 1$, 12, 23 . . . are taken each equal to $4/n^{\llcorner}$ and $Z_0 1'$, $1'2'$, $2'3'$ . . . also each equal to $4/n^{\llcorner}$ but measured in the contrary sense,

$$\overline{O1} = zi^{\frac{4}{n}},\ \overline{O2} = zi^{\frac{8}{n}} \dots \text{ and } \overline{O1'} = zi^{-\frac{4}{n}},\ \overline{O2'} = zi^{-\frac{8}{n}} \dots$$

so that

$$\overline{I1} = zi^{\frac{4}{n}} - 1, \overline{I2} = zi^{\frac{8}{n}} - 1 \dots \text{ and } \overline{I1'} = zi^{-\frac{4}{n}} - 1, \overline{I2'} = zi^{-\frac{8}{n}} - 1 \dots$$

Hence the equation of § 1 asserts that

$$\frac{n}{\overline{IZ}} = \frac{1}{\overline{IZ_0}} + \left(\frac{1}{\overline{I1}} + \frac{1}{\overline{I1'}}\right) + \left(\frac{1}{\overline{I2}} + \frac{1}{\overline{I2'}}\right) + \dots$$

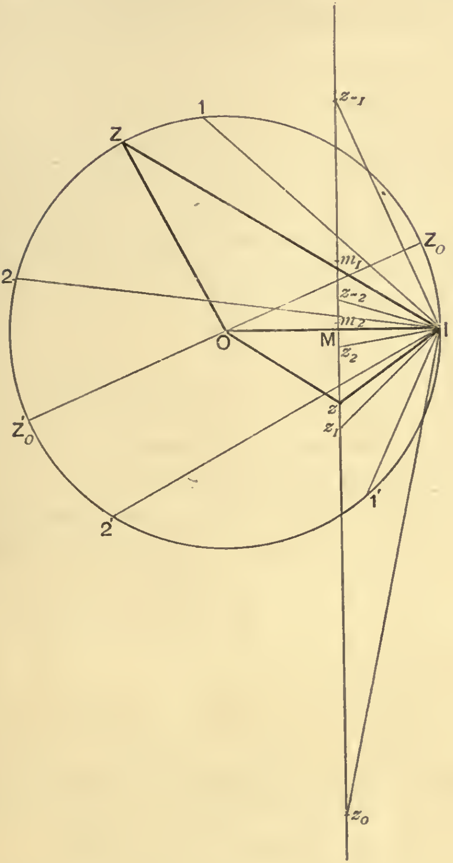
or that $\overline{IZ}$ is the vector harmonic mean of the n vector chords

$$\overline{IZ_0}, \overline{I1}, \overline{I1'}, \overline{I2}, \overline{I2'}, \dots \overline{I\left(\frac{n-1}{2}\right)}, \overline{I\left(\frac{n-1}{2}\right)'}$$

if n is an odd number, or of

$$\overline{IZ_0}, \overline{I1}, \overline{I1'}, \overline{I2}, \overline{I2'}, \dots \overline{I\left(\frac{n-2}{2}\right)}, \overline{I\left(\frac{n-2}{2}\right)'}, \overline{IZ_0'},$$

if n is even.



(The diagram is drawn for the case of $n=5$.)

Now if $\overline{Iz}$ is the reciprocal of IZ or $\overline{Iz} = 1/IZ$, z is found by constructing a triangle OIz on the opposite side of OI similar to ZIO: hence OIz is isosceles and z is a point on the line which bisects OI at right angles and the angle $OIz = OIZ = 1 - u$, so that

$$\overline{Iz} = \frac{1}{2}\operatorname{cosec} u \,.\, i^{-1-u} = -\frac{1}{2} - i \,.\, \frac{1}{2}\cot u.$$

Similarly if $\overline{Iz_0}$, $\overline{Iz_1}$, $\overline{Iz_{-1}}$, $\overline{Iz_2}$, $\overline{Iz_{-2}}$ . . . are the reciprocals respectively of $\overline{IZ_0}$, $\overline{I1}$, $\overline{I1'}$, $\overline{I2}$, $\overline{I2'}$ . . ., the points z_0, z_1, z_{-1}, z_2, z_{-2} . . . all lie on the line bisecting OI at right angles, which is the inverse of the circle with respect to I as the centre and IO as the radius of inversion, and

$$Iz_\lambda = \frac{1}{2}\operatorname{cosec}\frac{u+2\lambda}{n}\, i^{-1-\frac{4+2\lambda}{n}} = -\frac{1}{2} - i \,.\, \frac{1}{2}\cot\frac{u+2\lambda}{n}$$

where λ may have any one of the n values $0, 1, -1, 2, -2$ . . . Hence $\overline{Iz}$ is the vector mean of the vectors $\overline{Iz_0}$, $\overline{Iz_1}$, $\overline{Iz_{-1}}$ . . . and, M being the middle point of OI, Mz is the (arithmetic) mean of Mz_0, Mz_1, Mz_{-1} . . . or $\cot u$ is the mean of the n cotangents,

$$\cot\frac{u}{n},\ \cot\frac{u+2}{n},\ \cot\frac{u-2}{n},\ \cot\frac{u+4}{n},\ \cot\frac{u-4}{n}, \ldots$$

It should be observed that z_0z_1, z_0z_{-1}, z_1z_2, $z_{-1}z_{-2}$ . . . all* subtend at I angles $= \frac{2^{\llcorner}}{n}$.

Also if m_1, m_2, m_3 . . ., are respectively the middle points of z_1z_{-1}, z_2z_{-2}, . . .

so that $\overline{Iz_1} + \overline{Iz_{-1}} = 2\overline{Im_1}$, $\overline{Iz_2} + \overline{Iz_{-2}} = 2\overline{Im_2}$, . . .

$\overline{Iz}$ is the mean of $\overline{Iz_0}$, $2\overline{Im_1}$, $2\overline{Im_2}$, . . . and Mz that of Mz_0, $2Mm_1$, $2Mm_2$, . . . and it is easy to show that

---

* (In what sense does z_0z_{-1} in the diagram subtend the angle $\frac{2^{\llcorner}}{n}$?)

$$\overline{Mm_r} = -i\,\frac{\sin\frac{u}{n}\cdot\cos\frac{u}{n}}{\sin^2\frac{u}{n}-\sin^2\frac{2r}{n}},\quad \overline{z_r m_r} = \overline{m_r z_{-r}} = -i\,\frac{\sin\frac{2r}{n}\cdot\cos\frac{2r}{n}}{\sin^2\frac{u}{n}-\sin^2\frac{2r}{n}}.$$

If $u=2r$, the chord $\overline{Ir'}$ vanishes, and therefore $\overline{Iz_{-r}}$ is infinite, as is also $\overline{Iz}$.

The corresponding interpretation for the series for $1/z^n+1$ may be left to the student as an exercise.

5. *cot u and tan u as the sum of an infinite series of fractions.*

The series for $\cot u$ in § 3 may be written thus (taking n as an odd number) :—

$$\cot u = \frac{1}{u}\cdot\left(\frac{u}{n}\cot\frac{u}{n}\right) - \frac{1}{2-u}\cdot\left(\frac{2-u}{n}\cot\frac{2-u}{n}\right) - \dots$$

$$\dots - \frac{1}{n-1-u}\left(\frac{n-1-u}{n}\cdot\cot\frac{n-1-u}{n}\right)$$

$$+\frac{1}{2+u}\left(\frac{2+u}{n}\cdot\cot\frac{2+u}{n}\right)+\dots$$

$$\dots+\frac{1}{n-1+u}\left(\frac{n-1+u}{n}\cot\frac{n-1+u}{n}\right)$$

or

$$\cot u = \frac{1}{u}\left(\frac{u}{n}\cot\frac{u}{n}\right) - \frac{n\sin\frac{2u}{n}}{\left(n\sin\frac{2}{n}\right)^2-\left(n\sin\frac{u}{n}\right)^2} - \dots$$

$$-\frac{n\cdot\sin\frac{2u}{n}}{\left(n\cdot\sin\frac{n-1}{n}\right)^2-\left(n\sin\frac{u}{n}\right)^2}.$$

Suppose n to be increased without limit, then, the angles being measured in *right angles*,

$$\llcorner\left\{\frac{2r\pm u}{n}\cot\frac{2r\pm u}{n}\right\}=\llcorner\left\{\frac{2r\pm u}{n}\Big/\tan\frac{2r\pm u}{n}\right\}=\frac{2}{\pi},$$

as long as r is finite, so that for any finite number of terms, the successive terms of the first series above become

$$\frac{2}{\pi}\cdot\frac{1}{u},\quad-\frac{2}{\pi}\cdot\frac{1}{2-u},\ \frac{2}{\pi}\cdot\frac{1}{2+u},\quad-\frac{2}{\pi}\cdot\frac{1}{4-u},\ \frac{2}{\pi}\cdot\frac{1}{4+u}\cdots$$

Also, when r is so large as to be comparable with n,

$$\frac{2r\pm u}{n}\cdot\cot\frac{2r\pm u}{n}=\frac{2r}{n}\cdot\cot\frac{2r}{n}=\frac{2r}{n}\Big/\tan\frac{2r}{n}<\frac{2}{\pi},$$

($2r$ being less than n), so that the terms after the r^{th} are less than those of

$$\frac{2}{\pi}\left(-\frac{1}{2r-u}+\frac{1}{2r+u}-\frac{1}{2r+1-u}+\frac{1}{2r+1+u}-\ldots\right)$$

an alternating series of evanescent terms. Hence we conclude that, for all values of u, except only for $u=0$ or a multiple of $2^{\llcorner}$, in which case $\cot u$ and one of the fractions on the right-hand side become infinite, the equation

$$\cot u=\frac{2}{\pi}\left\{\frac{1}{u}-\frac{1}{2-u}+\frac{1}{2+u}-\frac{1}{4-u}+\frac{1}{4+u}-\frac{1}{6-u}+\frac{1}{6+u}+\ldots\right\}$$

is quantitatively true, the infinite series being convergent.

By combining the pairs of terms, or by proceeding to the limit, when $n=\infty$, from the second form above, we have

$$\cot u=\frac{2}{\pi}\left\{\frac{1}{u}-\frac{2u}{2^2-u^2}-\frac{2u}{4^2-u^2}-\frac{2u}{6^2-u^2}-\ldots\right\}$$

in which the infinite series may be easily shown to be *absolutely* convergent. The proof we leave to the student.

Using circular measure, and putting $\theta = \frac{\pi}{2} u$, these series become

$$\cot\theta = \frac{1}{\theta} - \frac{1}{\pi - \theta} + \frac{1}{\pi + \theta} - \frac{1}{2\pi - \theta} + \frac{1}{2\pi + \theta} - \dots$$

$$= \frac{1}{\theta} - \frac{2\theta}{\pi^2 - \theta^2} - \frac{2\theta}{2^2\pi^2 - \theta^2} - \frac{2\theta}{3^2\pi^2 - \theta^2} - \dots$$

Putting $1 - u$ for u, or $\frac{\pi}{2} - \theta$ for θ in the series for $\cot u$ or $\cot\theta$, we have, for all values of u except $u = (2r+1)$ right-angles,

$$\tan u = \frac{2}{\pi}\left(\frac{1}{1-u} - \frac{1}{1+u} + \frac{1}{3-u} - \frac{1}{3+u} + \dots\right)$$

$$= \frac{4u}{\pi}\left(\frac{1}{1-u^2} + \frac{1}{3^2 - u^2} + \frac{1}{5^2 - u^2} + \dots\right)$$

or

$$\tan\theta = \frac{1}{\frac{\pi}{2} - \theta} - \frac{1}{\frac{\pi}{2} + \theta} + \frac{1}{3\frac{\pi}{2} - \theta} - \frac{1}{3\frac{\pi}{2} + \theta} + \dots$$

$$= 2\theta\left\{\frac{1}{\left(\frac{\pi}{2}\right)^2 - \theta^2} + \frac{1}{\left(3\frac{\pi}{2}\right)^2 - \theta^2} + \frac{1}{\left(5\frac{\pi}{2}\right)^2 - \theta^2} + \dots\right\}$$

6. *Series for cot u deduced from the geometrical interpretation.*

Recurring to the geometrical interpretation, it will be seen that, as n increases without limit, the points z_0, 1, 1′, 2, 2′, . . . r, r', where r is any finite number, ultimately coincide with I, and therefore $z_0, z_1, z_{-1}, \dots z_r, z_{-r}$ are ultimately at infinity. But, since the arc $IZ_0 = \text{arc } IZ/n$,

$$\frac{Mz_0}{n} = \frac{Mz_0}{Iz_0} \cdot Iz_0 \cdot \frac{\text{arc } IZ_0}{\text{arc } IZ} = \frac{Mz_0}{Iz_0} \cdot \frac{\text{arc } IZ_0}{IZ_0} \cdot \frac{1}{\text{arc } IZ}$$

and

$$\llcorner \frac{Mz_0}{Iz_0} = 1, \quad \llcorner \frac{\text{arc } IZ_0}{\text{chord } IZ_0} = 1, \quad \text{arc } IZ = 2u \cdot \frac{\pi}{2}$$

therefore $\llcorner \frac{Mz_0}{n} = \frac{1}{\pi u} = \frac{1}{2\theta}$, if θ is the measure of $u^{\llcorner}$ in radians

Also

$$\frac{Mz_r}{n} = \frac{Mz_r}{Iz_r} \cdot Iz_r \cdot \frac{\text{arc } IZ_0}{\text{arc } IZ} = \frac{Mz_r}{Iz_r} \cdot \frac{\text{arc } IZ_0}{Ir} \cdot \frac{1}{\text{arc } IZ}$$

and

$$\llcorner \frac{Mz_r}{Iz_r} = 1, \llcorner \left(\frac{\text{arc } IZ_0}{\text{chd } Ir}\right) = \llcorner \left(\frac{\text{arc } IZ_0}{\text{arc } Ir} \cdot \frac{\text{arc } Ir}{\text{chd } Ir}\right) = \frac{2u}{2u+4r} \cdot 1$$

therefore

$$\llcorner \frac{Mz_r}{n} = \frac{1}{\pi(u+2r)} = \frac{1}{2(\theta + r\pi)}$$

and in like manner

$$\llcorner \frac{Mz_{-r}}{n} = \frac{1}{\pi(u-2r)} = \frac{1}{2(\theta - r\pi)}.$$

When however r becomes comparable with n, the chord Ir is finite, and therefore Iz_r and Mz_r are finite, and Mz_r, Mz_{-r} are ultimately equal and of contrary signs, since, when Z_0 coincides with I, $I1 = I1'$: hence $\llcorner (Mz_r + Mz_{-r}) = 0$, and still more is $\llcorner \frac{Mz_r + Mz_{-r}}{n} = 0$ and $\llcorner \Sigma \left(\frac{Mz_r}{n} + \frac{Mz_{-r}}{n}\right) = 0$ when n is infinite, and r less than, but comparable with, n. Hence, since $Mz = \frac{1}{2} \cot u$, and

$$Mz = \frac{Mz_0}{n} + \frac{Mz_1}{n} + \frac{Mz_{-1}}{n} + \ldots$$

$$\cot u = \frac{2}{\pi}\left\{\frac{1}{u} - \frac{1}{2-u} + \frac{1}{2+u} - \frac{1}{4-u} + \frac{1}{4+u} - \ldots\right\};$$

or $$\cot \theta = \frac{1}{\theta} - \frac{1}{\pi - \theta} + \frac{1}{\pi + \theta} - \frac{1}{2\pi - \theta} + \frac{1}{2\pi + \theta} - \ldots$$

7. *Cosec u and sec u as the sum of an infinite series of fractions.*

Since

$$\tan\frac{u}{2}+\cot\frac{u}{2}=\frac{\sin^2\frac{u}{2}+\cos^2\frac{u}{2}}{\sin\frac{u}{2}\cos\frac{u}{2}}=\frac{2}{\sin u}=2\operatorname{cosec} u,$$

and from § 5,

$$\frac{1}{2}\tan\frac{u}{2}=\frac{2}{\pi}\left\{\frac{1}{2-u}-\frac{1}{2+u}+\frac{1}{6-u}-\frac{1}{6+u}+\ldots\right\}$$

$$\frac{1}{2}\cot\frac{u}{2}=\frac{2}{\pi}\left\{\frac{1}{u}-\frac{1}{4-u}+\frac{1}{4+u}-\frac{1}{8-u}+\frac{1}{8+u}-\ldots\right\}$$

therefore

$$\operatorname{cosec} u=\frac{2}{\pi}\left\{\frac{1}{u}+\frac{1}{2-u}-\frac{1}{2+u}-\frac{1}{4-u}+\frac{1}{4+u}+\frac{1}{6-u}\right.$$
$$\left.-\frac{1}{6+u}-\frac{1}{8-u}+\frac{1}{8+u}+\ldots\right\}$$
$$=\frac{2}{\pi}\left\{\frac{1}{u}+\frac{2u}{2^2-u^2}-\frac{2u}{4^2-u^2}+\frac{2u}{6^2-u^2}-\ldots\right\}$$

and, putting $1-u$ for u,

$$\sec u=\frac{2}{\pi}\left\{\frac{1}{1-u}+\frac{1}{1+u}-\frac{1}{3-u}-\frac{1}{3+u}+\frac{1}{5-u}+\frac{1}{5+u}-\ldots\right\}$$
$$=\frac{4}{\pi}\left\{\frac{1}{1-u^2}-\frac{3}{3^2-u^2}+\frac{5}{5^2-u^2}-\ldots\right\}$$

In circular measure these series become

$$\operatorname{cosec}\theta=\frac{1}{\theta}+\frac{2\theta}{\pi^2-\theta^2}-\frac{2\theta}{(2\pi)^2-\theta^2}+\frac{2\theta}{(3\pi)^2-\theta^2}-\ldots$$

$$\sec\theta=\pi\left\{\frac{1}{\left(\frac{\pi}{2}\right)^2-\theta^2}-\frac{3}{\left(\frac{3\pi}{2}\right)^2-\theta^2}+\frac{5}{\left(\frac{5\pi}{2}\right)^2-\theta^2}-\ldots\right\}$$

[Deduce the following series for π:

$$(1)\quad \pi=3+\frac{6}{5\,.\,7}-\frac{6}{11\,.\,13}+\frac{6}{17\,.\,19}-\frac{6}{23\,.\,25}+\dots$$

$$(2)\quad \pi=4\,(\sqrt{2}-1)\left\{\frac{1}{2}-\frac{1}{7\,.\,9}-\frac{1}{15\,.\,17}-\frac{1}{23\,.\,25}-\dots\right\}$$

$$(3)\quad \pi=18\left\{\frac{1}{1\,.\,5}-\frac{3}{7\,.\,11}+\frac{5}{13\,.\,17}-\frac{7}{19\,.\,23}+\dots\right\}\Bigg]$$

8. *Excircular functions as the sum of an infinite series of fractions.*

By writing iu for u in the foregoing series, we obtain

$$\tanh u \;=-i\,.\tan(iu)=\frac{4u}{\pi}\left\{\frac{1}{1+u^2}+\frac{1}{3^2+u^2}+\frac{1}{5^2+u^2}+\dots\right\}$$

$$\coth u \;= i\,.\cot iu \qquad =\frac{4u}{\pi}\left\{\frac{1}{2u^2}+\frac{1}{2^2+u^2}+\frac{1}{4^2+u^2}+\dots\right\}$$

$$\operatorname{sech} u \;= \sec iu \qquad =\frac{4}{\pi}\left\{\frac{1}{1+u^2}-\frac{3}{3^2+u^2}+\frac{5}{5^2+u^2}-\dots\right\}$$

$$\operatorname{cosech} u = i\operatorname{cosec} iu \qquad =\frac{4u}{\pi}\left\{\frac{1}{2u^2}-\frac{1}{2^2+u^2}+\frac{1}{4^2+u^2}-\dots\right.$$

EXAMPLES.

1. Verify the periodic character of the fraction series for $\tan u$, $\cot u$, &c.

2. Prove that

$$\sec^2 u = \frac{4}{\pi^2}\left\{\frac{1}{(1-u)^2} + \frac{1}{(1+u)^2} + \frac{1}{(3-u)^2} + \frac{1}{(3+u)^2} + \ldots\right\}$$

$$\operatorname{cosec}^2 u = \frac{4}{\pi^2}\left\{\frac{1}{u^2} + \frac{1}{(2-u)^2} + \frac{1}{(2+u)^2} + \frac{1}{(3-u)^2} + \frac{1}{(3+u)^2} + \ldots\right\}$$

3. From the preceding results, verify the identity

$$\sec^2 u + \operatorname{cosec}^2 u = 4 \operatorname{cosec}^2 2u.$$

4. Also deduce the following:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$

$$1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \ldots = \frac{\pi^2}{9}.$$

$$1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{11^2} + \ldots = \frac{4\pi^2}{27}$$

5. Prove that

$$\frac{1}{(1-u)^3} - \frac{1}{(1+u)^3} + \frac{1}{(3-u)^3} - \frac{1}{(3+u)^3} + \ldots = \frac{\pi^3}{8} \tan u \sec^2 u$$

and thence that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \frac{\pi^3}{32}.$$

6. Prove that

$$\frac{\cot u}{1-u} = \frac{4}{\pi}\left\{\frac{1}{u(2-u)} + \frac{1}{(2+u)(4-u)} + \frac{1}{(4+u)(6-u)} + \ldots\right.$$

7. From the resolution of $\frac{x^m}{x^n - 1}$ into partial fractions, where n is an odd number and m an integer less than n, prove that, for the values $\pm\frac{1}{n}, \pm\frac{3}{n}, \ldots \pm\frac{n-2}{n}$ of p,

A. $$\frac{\cos pu}{\sin u}=\frac{1}{n}\cot\frac{u}{n}+\frac{\cos 2p}{n}\left(\cot\frac{2-u}{n}-\cot\frac{2+u}{n}\right)$$

$$-\frac{\cos 4p}{n}\left(\cot\frac{4-u}{n}-\cot\frac{4+u}{n}\right)+\dots$$

$$-(-1)^{\frac{n-1}{2}}\frac{\cos(\overline{n-1}\,p)}{n}\left(\cot\frac{\overline{n-1}-u}{n}-\cot\frac{\overline{n-1}+u}{n}\right).$$

B. $$\frac{\sin pu}{\sin u}=\frac{\sin 2p}{n}\left(\cot\frac{2-u}{n}+\cot\frac{2+u}{n}\right)$$

$$-\frac{\sin 4p}{n}\left(\cot\frac{4-u}{n}+\cot\frac{4+u}{n}\right)+\dots$$

$$-(-1)^{\frac{n-1}{2}}\frac{\sin \overline{n-1}\,p}{n}\left(\cot\frac{\overline{n-1}-u}{n}+\cot\frac{\overline{n-1}+u}{n}\right).$$

C. $$\frac{\cos pu}{\cos u}=\frac{\cos p}{n}\left(\cot\frac{1-u}{n}+\cot\frac{1+u}{n}\right)$$

$$-\frac{\cos 3p}{n}\left(\cot\frac{3-u}{n}+\cot\frac{3+u}{n}\right)+\dots$$

$$-(-1)^{\frac{n-1}{2}}\frac{\cos \overline{n-2}\,p}{n}\left(\cot\frac{n-2-u}{n}+\cot\frac{n-2+u}{n}\right)$$

$$+(-1)^{\frac{n-1}{2}}\frac{\cos np}{n}\cot\frac{n-u}{n}.$$

D. $$\frac{\sin pu}{\cos u}=\frac{\sin p}{n}\left(\cot\frac{1-u}{n}-\cot\frac{1+u}{n}\right)$$

$$-\frac{\sin 3p}{n}\left(\cot\frac{3-u}{n}-\cot\frac{3+u}{n}\right)+\dots$$

$$-(-1)^{\frac{n-1}{2}}\left(\cot\frac{n-2-u}{n}+\cot\frac{\overline{n-2}+u}{n}\right)+(-1)^{\frac{n-1}{2}}\frac{\sin np}{n}\cot\frac{2-u}{n}.$$

8. From the preceding results prove that, if $p^2<1$,

A. $$\frac{\cos pu}{\sin u}=\frac{4}{\pi}u\left\{\frac{1}{2u^2}+\frac{\cos 2p}{2^2-u^2}-\frac{\cos 4p}{4^2-u^2}+\frac{\cos 6p}{6^2-u^2}-\dots\right\}$$

$$B.\quad \frac{\sin pu}{\sin u}=\frac{8}{\pi}\left\{\frac{\sin 2p}{2^2-u^2}-\frac{2\sin 4p}{4^2-u^2}+\frac{3\sin 6p}{6^2-u^2}-\dots\right\}$$

$$C.\quad \frac{\cos pu}{\cos u}=\frac{4}{\pi}\left\{\frac{\cos p}{1-u^2}-\frac{3\cos 3p}{3^2-u^2}+\frac{5\cos 5p}{5^2-u^2}-\dots\right\}$$

$$D.\quad \frac{\sin pu}{\cos u}=\frac{4}{\pi}u\left\{\frac{\sin p}{1-u^2}-\frac{\sin 3p}{3^2-u^2}+\frac{\sin 5p}{5^2-u^2}-\dots\right\}$$

9. $$\operatorname{sech}^2 u=\frac{8}{\pi^2}\left\{\frac{1-u^2}{(1+u^2)^2}+\frac{3^2-u^2}{(3^2+u^2)^2}+\frac{5^2-u^2}{(5^2+u^2)^2}+\dots\right\}$$

10. If p is an integer $<n$,

$$\frac{n\cos^p\theta}{\cos n\theta}=\frac{\sin\frac{1}{n}\cos^p\frac{1}{n}}{\cos\theta-\cos\frac{1}{n}}-\frac{\sin\frac{3}{n}\cos^p\frac{3}{n}}{\cos\theta-\cos\frac{3}{n}}+\dots\pm\frac{\sin\frac{2n-1}{n}\cos^p\frac{2n-1}{n}}{\cos\theta-\cos\frac{2n-1}{n}}.$$

11. $$\frac{1}{e-1}=\frac{1}{2}+\frac{2}{2^2\pi^2+1}+\frac{2}{4^2\pi^2+1}+\frac{2}{6^2\pi^2+1}+\dots$$

$$\frac{1}{e+1}=\frac{1}{2}-\frac{2}{\pi^2+1}-\frac{2}{3^2\pi^2+1}-\frac{2}{5^2\pi^2+1}-\dots$$

12. $$\frac{1}{\eta-1}=-\frac{1}{2}+\frac{4}{\pi}\left\{\frac{1}{2}+\frac{1}{4^2+1}+\frac{1}{8^2+1}+\frac{1}{12^2+1}+\dots\right\}$$

$$\frac{1}{\eta^2-1}=-\frac{1}{2}+\frac{2}{\pi}\left\{\frac{1}{2}+\frac{1}{2^2+1}+\frac{1}{4^2+1}+\frac{1}{6^2+1}+\frac{1}{8^2+1}+\dots\right\}$$

13. From the formula $\cot u=\frac{1}{2}\left(\cot\frac{u}{2}+\cot\frac{u+2}{2}\right)$ deduce that, if $m=2^{n-1}$,

$$\cot u=\frac{1}{2m}\left\{\cot\frac{u}{2m}+\Sigma_1^{m-1}\left(\cot\frac{2r+u}{2m}-\cot\frac{2r-u}{2m}\right)+\cot\frac{u+2m}{2m}\right\}$$

and thence the fraction series for $\cot u$. (Schröter, *Schlömilch's Zeitschrift*, 1868.)

CHAPTER XII

RATIONAL AND INTEGRAL FUNCTIONS

THE general theory of functions of a complex variable forms one of the most important of the more recent developments of Mathematics, whether regarded with reference to abstract theory or to its applications to Mathematical Physics. This theory in its general aspect lies beyond the scope of an elementary work like the present; but it may prove useful to the student, as introductory to the wider theory, to consider here a few general properties of that particular class of functions of a complex variable, known as Rational and Integral Functions.

1. *Rational and Integral Functions—Definition, Notation, &c.*

Let z denote a vector or complex number, whose tensor or modulus is r and versor i^u, and whose project is x and traject y, so that $z \equiv ri^u \equiv x + iy$: and let

$$w \equiv fz \equiv z^n + a_1 i^{\alpha_1} z^{n-1} + a_2 i^{\alpha_2} z^{n-2} + \dots + a_{n-1} i^{\alpha_{n-1}} z + a^n i^{\alpha_n}$$

where $a_1, a_2 \dots a_n$ are tensors or moduli, and $i^{\alpha_1} \dots i^{\alpha_n}$ versors, independent of z, and n is a finite positive integer: then fz is said to be a rational and integral function of z of the n^{th} degree.

Let ρ, i^v be the tensor or modulus and versor of fz, and X, Y its project and traject respectively, then

$$w \equiv fz \equiv \rho i^v \equiv X + iY$$

and we have the following identities:—

$$w \equiv \rho i^v \equiv X + iY \equiv r^n i^{nu} + a_1 r^{n-1} i^{(n-1)u+\alpha_1}$$
$$+ a_2 r^{n-2} i^{(n-2)u+\alpha_2} - \dots + a_n i^{\alpha_n}$$

$$X \equiv r^n \cos nu + a_1 r^{n-1} \cos(\overline{n-1}u + \alpha_1) \\ + a_2 r^{n-2} \cos(\overline{n-2}u + \alpha_2) - \dots + a_n \cos \alpha_n$$

$$Y \equiv r^n \sin nu + a_1 r^{n-1} \sin(\overline{n-1}\,u + \alpha_1) \\ + a_2 r^{n-1} \sin(\overline{n-2}\,u + \alpha_2) \dots + a_n \sin \alpha_n\,;$$

as also, 1_n, 2_n, $3_n, \dots (m)_n \dots$ denoting the coefficients of the expansion of $(1 + x)^n$.

$$r^p \cos pu \equiv x^p - 2_p x^{p-2} y^2 + 4_p x^{p-4} y - \dots$$

$$r^p \sin pu \equiv y \left\{1_p x^{p-1} - 3_p x^{p-3} y^2 + 5_p x^{p-5} y^4 - \dots\right\}$$

Whence X, Y can be expressed, if necessary, in terms of x, y.

2. *fz expressed in powers of* $z - z_0$.

Let z_0 be a particular value of z: then, supposing for simplicity the versor factors i^α to be included in the a coefficients,

$$\begin{aligned} fz - fz_0 &= (z^n - z_0{}^n) + a_1 (z^{n-1} - z_0{}^{n-1}) + a_2 (z^{n-2} - z_0{}^{n-2} \dots \\ &\qquad + a_{n-1}(z - z_0) \\ &= (z - z_0)\left\{(z^{n-1} + z_0 z^{n-2} + z_0{}^2 z^{n-3} + \dots + z_0{}^{n-1}) \right. \\ &\qquad + a_1 (z^{n-2} + z_0 z^{n-3} + \dots + z_0{}^{n-2}) + \dots \\ &\qquad \left. + a_{n-2}(z + z_0) + a_{n-1}\right\} \\ &= (z - z_0) f_1 z \text{ suppose,} \end{aligned}$$

where $f_1 z$ is a rational and integral function of the $\overline{n-1}^{\text{th}}$ degree. Arranging the expression for $f_1 z$ above according to powers of z,

$$\begin{aligned} f_1 z = z^{n-1} &+ (z_0 + a_1) z^{n-2} + (z_0{}^2 + a_1 z_0 + a_2) z^{n-3} \\ &+ (z_0{}^3 + a_1 z_0{}^2 + a_2 z_0 + a_3) z^{n-4} + \dots \\ &+ (z_0{}^{n-1} + a_1 z_0{}^{n-2} + a_2 z_0{}^{n-3} \dots + a_{n-2} z_0 + a_{n-1}) \end{aligned}$$

in which it is obvious that the coefficient of the term involving z^{n-r} is obtained from that of the preceding term by multiplying it by z_0 and adding the term a_{r-1}.

Thus it appears that $f_1 z$ is the quotient, and fz_0 the remainder when fz is divided by $z - z_0$, and a simple mode of calculating these has been found.

In like manner if $f_2z, f_3z, \ldots f_nz$ denote respectively the quotients, when $f_1z, f_2z, \ldots f_{n-1}z$ are divided by $z - z_0$, so that they are functions of the degrees $n - 2, n - 3, \ldots 1, 0$ respectively, we have

$$\begin{aligned} fz &= (z - z_0)f_1z + fz_0 \\ f_1z &= (z - z_0)f_2z + f_1z_0 \\ f_2z &= (z - z_0)f_3z + f_2z_0 \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ f_{n-1}z &= (z - z_0)f_nz + f_{n-1}z_0 \\ f_nz &= 1, \end{aligned}$$

whence, multiplying both sides of the successive equations by $1, z - z_0, (z - z_0)^2, \ldots (z - z_0)^n$ respectively, and omitting like terms on opposite sides, we have

$$fz = (z - z_0)^n + f_{n-1}z_0\,(z - z_0)^{n-1} + f_{n-2}z_0\,(z - z_0)^{n-2} \ldots \\ + f_1z_0\,(z - z_0) + fz_0$$

or, if we put h for $z - z_0$,

$$f(z_0 + h) = fz_0 + f_1z_0 \,.\, h + f_2z_0 \,.\, h^2 + \ldots + f_{n-1}z_0 \,.\, h^{n-1} + f_nz_0 \,.\, h^n$$

[3. The functions* $f_1z_0, f_2z_0, \ldots$ may be obtained in their general form by putting $z_0 + h$ for z in fz, expanding each term by the Binomial Theorem, and arranging the resulting expression according to powers of h. Hence it will be found that

$$f_p(z_0) = (p)_n z_0^{n-p} - (p)_{n-1}a_1z_0^{n-p-1} + (p)_{n-2}a_2z_0^{n-p-2} \ldots \\ + (-1)^{n-p}a_{n-p}.$$

The previous investigation, however, suggests a far simpler mode of calculating the values of $f_1z_0, f_2z_0 \ldots$ in any particular case than by substituting in the general form, especially when the coefficients are numerical. This will be best shown in an example.

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* The student, familiar with the elements of the Differential Calculus, will also recognise $f_1z_0, f_2z_0, \ldots$ as the values for $z = z_0$ of the successive *divided* differential coefficients or derived functions of fz: or that $f_1z_0 = f'z_0$, $f_2z_0 = \dfrac{f''z_0}{1\,.\,2}$, $f_3z_0 = \dfrac{f'''z_0}{1\,.\,2\,.\,3}, \ldots$

Take $fz = z^4 - a_1z^3 + a_2z^2 - a_3z + a_4$: then the work stands thus :—

$$\begin{array}{llllll}
1 & -a_1 & +a_2 & -a_3 & +a_4 & \\
1 & z_0 - a_1 & z_0^2 - a_1z_0 + a_2 & z_0^3 - a_1z_0^2 + a^2z_0 - a_3 & z_0^4 - a_1z_0^3 + a_2z_0^2 - a_3z_0 + a_4 = fz_0, & \\
1 & 2z_0 - a_1 & 3z_0^2 - 2a_1z_0 + a_2 & 4z_0^3 - 3a_1z_0^2 + 2a_2z_0 - a_3 = f_1z_0 & & \\
1 & 3z_0 - a_1 & 6z_0^2 - 3a_1z_0 + a_2 = f_2z_0 & & & \\
1 & 4z_0 - a_1 = f_3z_0 & & & & \\
1 = f_4z_0 & & & & &
\end{array}$$

each term being formed by adding to the term above it in the same column the product of z_0 and the preceding term in the same line. The first line consists of the coefficients of the terms of fz: the second gives the coefficients of the quotient and the remainder of $f_1z/(z-z_0)$: and so on.

For a numerical instance take $fz = z^4 - 2z^3 + 3z^2 - 4z + 5$: then to calculate $f(z+2)$ the work stands thus:

$$\begin{array}{ll}
1 - 2 + 3 - 4 + 5 & \\
1 \quad 0 + 3 + 2 + 9 & f(2) = 9 \\
1 + 2 + 7 + 16 & f_1(2) = 16 \\
1 + 4 + 15 & f_2(2) = 15 \\
1 + 6 & f_3(2) = 6 \\
1 & f_4(2) = 1
\end{array}$$

$\therefore f(z+2) = z^4 + 6z^3 + 15z^2 + 16z + 9.$

It is surprising that the above process, depending as it does on the simplest algebraical operations, and forming the basis of Horner's process for approximating to the roots of numerical equations, has not found a prominent place in elementary works on algebra. It is not too much to say that it is the natural and most useful sequel of the elementary operations. It should be called *Compound Involution.*]

4. *Continuity of* w. *An equation of the* n^{th} *degree has* n *roots.*

The functions $fz, f_1z, f_2z \ldots f_nz$ being rational and integral cannot be infinite for any finite value of z: hence from the expression

$$f(z+h) = fz + f_1z \, . \, h + f_2z \, . \, h^2 + \ldots + f_nz \, . \, h^n,$$

it follows that by taking the modulus of h sufficiently small, $f(z+h)$ may be made as near as we please to fz, becoming equal to fz when h vanishes, whatever finite value be given to z, and the same is for a like reason true for $f_1z, f_2z, \ldots$: therefore all these functions are *continuous* for all finite values of z.

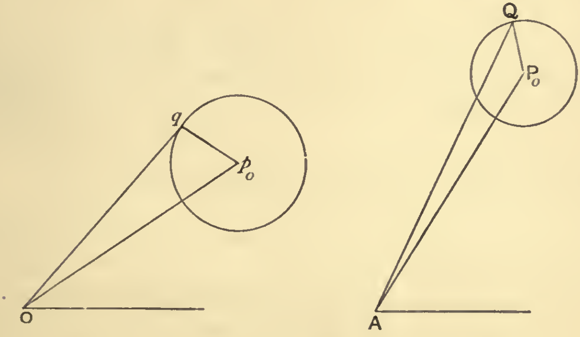
Also for the particular value z_0 of z, if f_1z_0 does not vanish, the terms which follow $f_1z_0 \, . \, h$ may be made by taking h sufficiently small, though still finite, less in any ratio than $f_1z_0 \, . \, h$, so that for very small values of h we may take

$$f(z_0 + h) = fz_0 + f_1z_0 \, . \, h \text{ very nearly.}$$

If $f_1 z_0$ vanishes, but $f_2 z_0$ does not, then in like manner, when h is very small, $f(z_0 + h) = fz_0 + f_2 z_0 . h^2$ very nearly; and generally if $f_1 z_0, f_2 z_0, \ldots f_{m-1} z_0$ all vanish, but $f_m z_0$ does not, then, when h is very small, $f(z_0 + h) = fz_0 + f_m z_0 . h^m$ very nearly.

Now fz, involving only integral powers of z, is a single-valued function, and thus corresponding to every value of z, there is one and only one value of fz or w. We proceed to show conversely that corresponding to any assumed value of w, there is at least one value of z, and that there are, in fact, n values of z.

Take two planes, in one of which (the z plane) the vector $\overline{Op} = z$, while in the other (the w plane) the vector $\overline{AP} = w$ or fz, so that for every point p in the z plane there is a corresponding point P in the w plane, and only one such point. Then at the particular point p_0, if $Op_0 = z_0$, for the corresponding point



P_0, $\overline{AP_0} = w_0$ or fz_0. Let q be a point, near to p_0, such that $\overline{p_0 q} = h = ki^{\kappa}$, and therefore $oq = z_0 + h = r_0 i^{\theta_0} + ki^{\kappa}$, then Q corresponding to q, if $f_1 z_0$ does not vanish, $\overline{AQ} = f(z_0 + h) = fz_0 + f_1 z_0 . h$ very nearly, when h is small, and $\overline{AP_0} = fz_0$, therefore $\overline{P_0 Q} = f_1 z_0 . ki^{\kappa}$ very nearly. Hence, if q moves round a small circle, centre p_0 and radius k, Q moves round a corresponding curve very nearly a small circle, centre P_0 and radius

$= k \,.\, \text{mod}\,(f_1 z_0)$, and *one* complete circuit of q determines one complete circuit of Q.

If $f_1 z_0$ vanishes, but $f_2 z_0$ does not, $\overline{P_0 Q} = f_2 z_0 \,.\, k^2 i^{2\kappa}$ nearly, and then an increase of κ by 4, giving an increase of 2κ by 8, one complete circuit of q determines *two* complete circuits of Q. So generally if $f_1 z_0, f_2 z_0, \,.\,.\,.\, f_{m-1} z_0$ all vanish, but $f_m z_0$ does not, $\overline{P_0 Q} = f_m z_0 \,.\, k^m i^{m\kappa}$, and one complete circuit of q determines m complete circuits of Q.

Hence it follows that for every point on the w plane there is a corresponding point on the z plane. For if there were any region on the w plane, the points in which had no corresponding points in the z plane, P_0 being taken on or close to the boundary of such region, Q in moving round P_0, while q moves round p_0, would pass into that region, and thus there would be points in it corresponding to points in the z plane, contrary to the hypothesis. Thus for any assumed value of w or fz, there is certainly *a* value of z.

Now for the origin A $w = 0$, and as we have seen there must be some point in the z plane corresponding to A, therefore there is some value of z, for which $fz = 0$; and this proves that

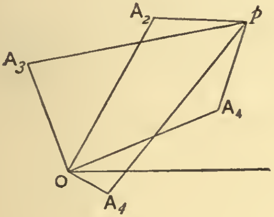
"*Every rational and integral equation has a root.*"

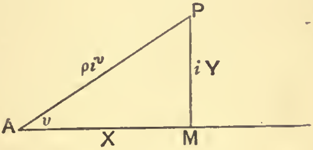
Let then z_1 denote such a root, so that $fz_1 = 0$, then fz contains the factor $z - z_1$, and the other factor, the quotient of fz divided by $z - z_1$, will be a rational and integral function of the $\overline{n-1}^{\text{th}}$ degree. For a like reason this last function will have a factor $z - z_2$ and its second factor will be of the $\overline{n-2}^{\text{th}}$ degree: and this again will have a factor $z - z_3$ and another factor of the $\overline{n-3}^{\text{th}}$ degree, and so on. Hence finally fz must be the product of n factors of the form $z - z_1$, or

$$fz = (z - z_1)(z - z_2) \,.\,.\,.\, (z - z_n),$$

from which it is evident that fz vanishes, for the n values z_1, $z_2 \,.\,.\,.\, z_n$ and for these only. Hence, understanding that when two or more of these values are equal, each still counts as one root, "*Every rational and integral equation of the* n$^{\text{th}}$ *degree has* n *roots.*"

Let $\overline{OA}_1$, $\overline{OA}_2 \ldots \overline{OA}_n$ be the n vectors in the z plane corresponding to the values $z_1, z_2, \ldots z_n$, then $A_1, A_2, \ldots A_n$ are the n points on the z plane corresponding to the origin A on the w plane. They are termed the *radical points* for the function fz. In the case where two or more of the roots are equal, the corresponding number of radical points coincide, and the point of coincidence must be regarded as having a *weight* equal to the number of coincident points. Also for any point P_0 in the w plane such that $\overline{AP}_0 = w_0$, the equation $fz = w_0$ being of the n^{th} degree in z has n roots, so that to any point on the w plane there correspond n points on the z plane, the radical points of the function $fz - w_0$.





5. *Expression of* w *in vector factors.*

Let p be any point in the z plane and $\overline{Op} = z$, then

$$z - z_1 = \overline{Op} - \overline{OA}_1 = \overline{A_1 p}$$

and similarly $\quad z - z_2 = \overline{A_2 p},\ \overline{z - z_3} = A_3 p \ldots,$

and therefore

$$w = fz = \overline{A_1 p} \cdot \overline{A_2 p} \cdot \overline{A_3 p} \ldots \overline{A_n p}$$

whence the points $A_1, A_2, \ldots A_n$ being given, the value of w or AP corresponding to any point p may be determined, and it is to be observed that this determination is quite independent of the origin O, which may therefore be changed arbitrarily without altering w, though as such change alters z, it also alters the form of fz in respect of its coefficients. It will often be convenient to assume the origin to be the mean point of the n radical points, in which case

$$\overline{OA_1}+\overline{OA_2}+\ldots+\overline{OA_n} \text{ or } z_1+z_2+\ldots+z_n=0.$$

In this case the coefficient of z^{n-1} in fz, being (as appears by multiplying out the factors $z-z_1, z-z_2\ldots$) $z_1+z_2+\ldots+z_n$, will vanish, or the equation $fz=0$ will be deprived of its second term.

Let $\overline{A_1p}=\rho_1 i^{u_1}$, $\overline{A_2p}=\rho_2 i^{u_2}\ldots A_np=\rho_n i^{u_n}$,

then we have

$$w=\rho i^{v}=\rho_1\rho_2\ldots\rho_n i^{u_1+u_2+\ldots+u_n},$$

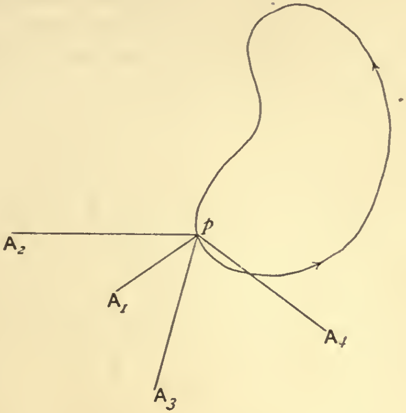
whence $\rho=\rho_1\rho_2\ldots\rho_n$ and $v=u_1+u_2\ldots+u_n$

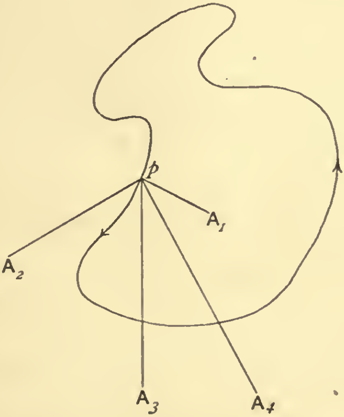
or the tensor of $\overline{AP}$ or w is the product of the tensors of $\overline{A_1p_1}\ldots\overline{A_np}$, and its inclination the sum of their inclinations.

6. *Tests for the number of roots within given limits. Cauchy's Theorem.*

Take any closed curve in the z plane, and let the point p describe this contour always in the *positive* sense, which we shall take to be that in which a man advancing would have the inside of the contour on his left, as shown by the arrow-heads in the figures. Then leaving out of account the variations of $\rho_1\,\rho_2\ldots\rho_n$ and ρ, consider the variations of $u_1\,u_2\ldots u_n$ and the consequent variation of v. First, suppose that the contour includes none of the radical points A_1, $A_2\ldots$ Then it is plain that in a complete circuit the amounts of positive and negative rotation of A_1p about A_1 exactly balance one another, so that at the end u_1 remains unchanged. The same will for like reasons be true of $u_2, u_3\ldots u_n$, and therefore also for v, which is equal to their sum

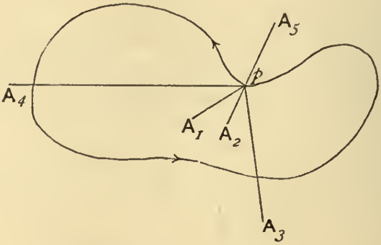
Next, suppose the contour to include the radical point A_1 and none of the rest. Then at the end of a complete circuit of p, the balance of positive and negative rotations of $A_1 p$ about A_1 will give an increase of u_1 by $4^{\llcorner}$, while $u_2\, u_3 \ldots u_n$ remain unchanged :





therefore v will have increased by $4^{\llcorner}$, or AP will have made one complete revolution round A.

Again suppose the contour to include two radical points A_1, A_2 and no others. Then by like reasoning both u_1 and u_2 are increased by $4^{\llcorner}$ and therefore v by eight right angles, or AP will have made *two* complete revolutions about A. The same will be true, however near A_1 is to A_2 and therefore also if A_1 and A_2 coincide, as they do when the two roots z_1, z_2 are equal.*



Lastly, similar reasoning proves generally that if the contour includes m radical points, m being the sum of the weights of the points where several are coincident, for one complete circuit of p there will be an increase of v by $4m^{\llcorner}$, or AP will make m complete revolutions about A.

Hence *the number of complete revolutions of AP about A, when* p *makes one complete circuit round any closed curve in the* z *plane, is the same as, and so determines, the number of radical points within that contour.*

From the identity $\rho i^v = X + iY$ in § 1, we have $\tan v = \frac{Y}{X}$, so that if the value of Y/X were determined for each point on the contour the changes of v would be completely determined. Now if, as

---

* This may also be seen from the consideration that in this case fz has the factor $(z-z_1)^2$ and therefore both fz_1 and $f_1 z_1$ vanish, so that

$$f(z_1+h)=f_2 z_1 h^2+f_3 z_1 h^3+\ldots - = f_2 z_1 h^2 \text{ very nearly, when } h \text{ is small,}$$

so that for one complete circuit of a *small* contour round A_1 there are two complete revolutions of AP about A. Then it may be shown that for any finite contour including no other radical points, the number of complete revolutions is the same.

v increases or AP revolves in the positive sense, $\tan v$ changes sign through 0, it changes from $-$ to $+$; while if, as v decreases or AP revolves in the negative sense, $\tan v$ changes sign through 0, it changes from $+$ to $-$, and in one complete revolution it makes in either case *two* such changes. Hence in making one complete revolution irregularly, that is, where the forward rotation exceeds the backward rotation by one whole revolution, the number of changes of sign through 0 of $\tan v$ from $-$ to $+$ must exceed those from $+$ to $-$ by two: and so in making m complete revolutions the first must exceed the other by $2m$.

Hence *if in one complete circuit of* p, Y/X* *changes sign by the vanishing of* Y k *times from* $-$ *to* $+$ *and* l *times from* $+$ *to* $-$ $\frac{1}{2}(k-l)$ *will be the number of complete revolutions of* AP *and therefore the number of radical points within the contour described by* p.

This result is due to Cauchy, but the proof above will (it is believed) be found easier and more direct than that given by him.

7. *Conjugate Functions.*

1. A function of a complex variable $x + iy$ may always be reduced to the form $X + iY$, where X, Y are scalar functions of the scalars x, y. The functions X, Y have important properties in relation to one another, from which they are called *Conjugate Functions*, and the theory of such functions has wide and important applications in the investigations of Mathematical Physics. This is not the place to treat of the Theory of Conjugate Functions generally, but some of its principal propositions may here be exemplified in the case of Rational and Integral Functions.

We have seen that by taking for the origin in the z plane the mean point of the n radical points, fz is changed as to its coefficients, so that the coefficient a_1 vanishes while w is unaltered hence with this origin we have from § 1,

---

* If there be no radical point on the contour, X and Y cannot both vanish for the same position of p.

$$X = r^n \cos nu + a_2 r^{n-2} \cos (\overline{n - 2}u + a_2) + \ldots + a_n \cos a_n$$

$$Y = r^n \sin nu + a_2 r^{n-2} \sin (\overline{n - 2}u + a_2) + \ldots + a_n \sin a_n.$$

Consider the curves on the z plane corresponding to the lines parallel to the axes on the w planes, for which the conjugate functions X, Y are constant. Then since the values of X, Y are unique for each point on the plane, no two curves of the series $X = X_0$, or of the series $Y = Y_0$, can intersect. But the curves corresponding to $X = 0$, $Y = 0$ respectively, must determine by their intersection the n radical points and no other; and so the curves corresponding to $X = X_0$, $Y = Y_0$ must determine by their intersection the n points corresponding to the extremity of the vector $X_0 + iY_0$. Hence, remembering that we are dealing only with *scalar* values of x, y and X, Y, any curve of the system $X =$ const. intersects any curve of the system $Y =$ const., though both are of the n^{th} degree, in n points and n only, reckoning 2, 3 . . . coincident points as 2, 3 . . . distinct points.

Also for the curve $X = X_0$, since

$$\cos nu = -\frac{a_2}{r^2} \cos (\overline{n - 2}\, u + a_1) - \ldots - \frac{a_n}{r^n} \cos a_n + \frac{X_0}{r^n},$$

when r is very large, $\cos nu$ is very small, and ultimately $\cos nu = 0$, when $r = \infty$, or $u = \frac{2\lambda + 1^{\llcorner}}{n}$, where λ is an integer; hence the curve has its ultimate direction, when r becomes infinite, that determined by one or other of the series of n angles, $\frac{1}{n}$, $\frac{3}{n}$ . . . $\frac{2n-1}{n}$, $\frac{2n+1}{n}$ . . . $\frac{4n-1}{n}$ right angles. Farther, since

$$r \cos nu = -\frac{a_2}{r} \cos (\overline{n - 2}\, u + a_1) \ldots - \frac{u_n}{r^{n-1}} \cos a_n + \frac{X_0}{r^{n-1}},$$

$\text{Lt}_{r=\infty} (r \cos nu) = 0$, from which it is to be inferred that these ultimate directions all pass through the origin. Hence the curves of the system $X = X_0$ have n rectilinear asymptotes all passing through the mean point of the n radical points, each inclined to the preceding at an angle of $2/n^{\llcorner}$, and the first making the angle $1/n^{\llcorner}$ with the prime vector.

Like reasoning proves that the curves of the system $Y = Y_0$ have n rectilinear asymptotes all passing through the mean point of the n radical points, each inclined to the preceding at an angle of $2/n^{\llcorner}$, and the first coincident with the prime vector.

In fact the asymptotes of either system are the bisectors of the angles between those of the other, as in the figure (for the case $n = 5$) where the continuous lines are the asymptotes of the Y system, and the broken lines those of the X system.

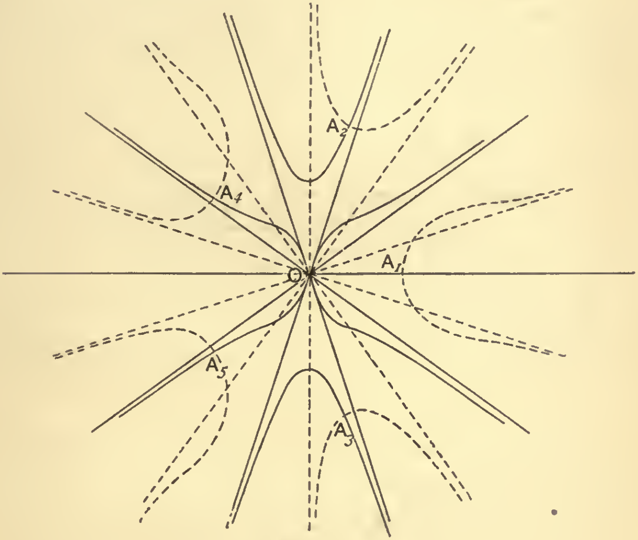


FIG. 1.

[The figure is drawn for the case, where $fz = z^5 + z^3 - 2$. The continuous lines represent the curve $Y = 0$ and its asymptotes, the prime vector axis OA_1 being reckoned as one of its branches. The dotted lines represent the curve $X = 0$ and its asymptotes. Their intersections A_1, A_2, A_3, A_4, A_5 are the 5 radical points. The student should verify these results.]

Now consider the curves in the neighbourhood of the ordinary point p_0, that is, a point for which $f_1 z_0$ does not vanish. We have seen that, if a circle, rad. k, is described about p_0 as a centre, the corresponding curve on the w plane, when k is very small, is a circle, rad. k mod. $f_1 z_0$, about P_0 as a centre, and one circuit of the one circle corresponds to one circuit of the other, so that the angles between corresponding radii in the two circles are equal.

Hence since the lines $X = X_0$, $Y = Y_0$ on the w plane intersect at P_0 at right angles, the corresponding lines on the z plane passing through the point p_0 intersect also at right angles. Therefore the curves $X = X_0$, $Y = Y_0$ on the z plane, form systems orthogonal to one another, or any curve of the one system cuts any curve of the other at right angles.

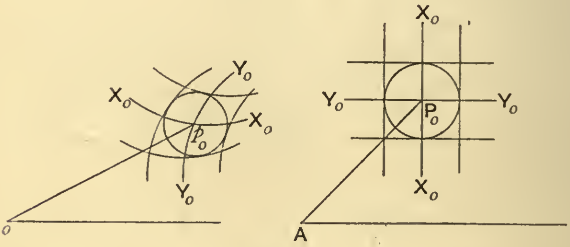
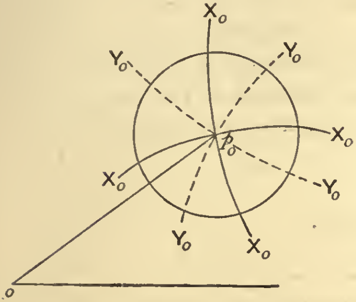
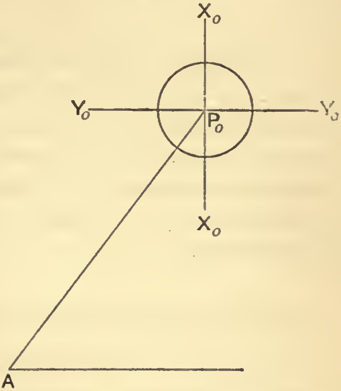


FIG. 2.

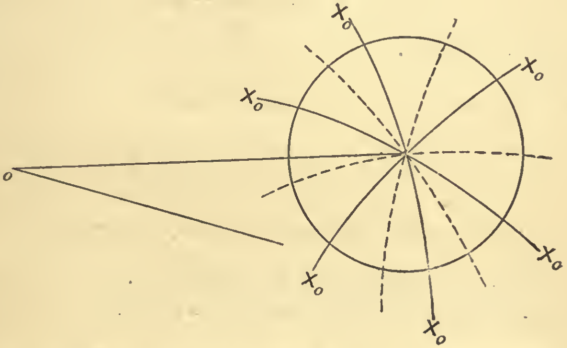
It results from this that if the w plane is divided up into elementary squares by the straight lines $X = X_0$, $Y = Y_0$ drawn at small equal intervals, the z plane will be divided by the corresponding curves into elementary areas which are ultimately squares, when the intervals are sufficiently small, and the area of any element in the w plane is to the corresponding element in the z plane in the ratio of $(\text{mod.}\, f_1 z_0)^2$ to 1.

There are $n - 1$ points, for which $f_1 z = 0$, and at these points, as we have seen above, the circles about P_0 and p_0 correspond, but in such a way that one circuit of the circle about p_0 corresponds to two about P_0, so that the angle between two radii of the p circle is only half that between the corresponding radii of the P circle. Hence the curve corresponding to $X = X_0$ has two branches passing through p_0 at right angles to one another, and that to $Y = Y_0$ has also two branches bisecting the angles between the former.





If two of these $\overline{n-1}$ points coincide, at this point f_2z vanishes as well as f_1z, and then one circuit of the circle about p_0 corresponds to *three* about P_0, so that the angle between two radii



of the p circle is only one-third of that between the corresponding radii of the P_0 circle, and both $X = X_0$ and $Y = Y_0$ have three branches passing through the point.

So generally if r of these $\overline{n-1}$ points coincide, at such a point $f_1z, f_2z, \ldots f_rz$ all vanish, and there are $\overline{r+1}$ branches of both curves $X = X_0$ and $Y = Y_0$ passing through the point p_0.

The foregoing form the principal general properties of the curves representing the Conjugate Functions derived from a rational and integral function of a complex variable. Subject to these, the forms of the curves are infinitely varied according to the different values of the coefficients $a_2, a_3 \ldots a_n$. We will illustrate this by the case of a function of the third degree or cubic function.

8. *Discussion of a Cubic Function.*

We will limit the discussion to the case of a cubic function whose coefficients are *scalar*, and as we have seen that there is no loss of generality by assuming the coefficient of the second term to be zero, since this only amounts to referring the curve systems to the mean of the radical points as origin, we may write

$$w = fz = z^3 + az + b,$$

where a, b are scalars.

Hence the values of the conjugate functions X, Y are

$$X = x^3 - 3xy^2 + ax + b \qquad Y = 3x^2y - y^3 + ay$$

or

$$X - b = x(x^2 - 3y^2 + a) \qquad Y = y(3x^2 - y^2 + a),$$

and we have to consider the characters of the curve systems $X = X_0$, $Y = Y_0$ for different values of a and b.

We observe that the asymptotes of either system are the same for all the curves of that system, and are independent of the values of a and b. They are for the X system the axis of y, and two lines through the origin inclined to it at angles of 60°

on either side, and for the Y system the axis of x and two lines through the origin inclined to it at angles of 60° on either side, so that the asymptotes of one system bisect the angles between the asymptotes of the other. Also the Y system is a system of curves of the third degree, which is completely determinate when a is given, and the same may be said of the X' system, if we put X' for $X - b$. In the particular case of $Y = 0$, the curve of the third degree becomes the axis of x, and an hyperbola whose asymptotes are the inclined asymptotes of the Y system; and in that of $X' = 0$ or $X = b$, the curve becomes the axis of y and an hyperbola, whose asymptotes are the inclined asymptotes of the X system. These form guiding curves or curved axes of the two systems.

Farther, since $f_1 z = 3z^2 + a$, $f_1 z$ will vanish, if $z^2 = -\frac{a}{3}$; that is, if a is positive, for the two points $x = 0$, $y = \pm\sqrt{\frac{a}{3}}$; but, if a is negative, for the two points $x = \pm\sqrt{-\frac{a}{3}}$, $y = 0$. These, then, will be *double* points on the curves of both systems, which pass through them, and there are only two such points in either system. The curves through the double points will form further guiding curves of the system. In the special case, where $a = 0$, the two double points merge in one and produce a *triple point* at the origin.

I. Suppose a to be positive, and put $3k^2$ for a. Then the double points are $x = 0$, $y = \pm k$, and the curves passing through them are $X = b$, $Y = \pm 2k^3$. The curve $X = b$ or $x(x^2 - 3y^2 + 3k^2) = 0$, has for its three branches the axis of y, and the two branches of the hyperbola, whose vertices are the points $\pm k$ (B, B' in the figure) on that axis, the semi-axes being k and $k\sqrt{3}$, as shown in the figure by the continuous lines.

If $X < b$, it will be easily seen that the equation $x(x^2 - 3y^2 + 3k^2)$ = a negative constant defines a curve which has three branches, such as those shown by the broken lines (P_1, P_2, P_3) in the figure; while, if $X > b$, the curve will have three branches, such as those shown by the dotted line (Q_1, Q_2, Q_3) in the figure.

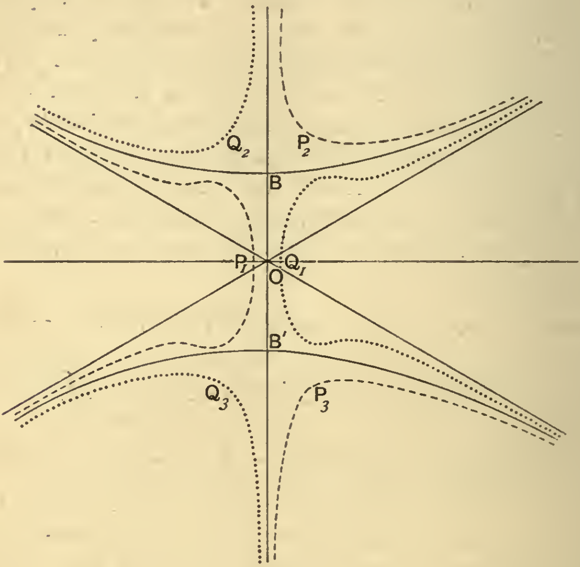


FIG. 1.

The continuous change from one form to the other, as X passes through the value b, is evident also from the figure.

The curve $Y = 2k^3$ or $y(3x^2 - y^2 + 3k^2) = 2k^3$ reduces to the form

$$3x^2y - (y-k)^2(y+2k) = 0.$$

Hence, since $\operatorname*{limit}_{x=0}\left(\frac{\overline{y-k}^{\,2}}{x^2}\right) = \operatorname*{limit}_{y=k}\left(\frac{3y}{y+2k}\right) = 1$, the two branches passing through the double point B ($x = 0$, $y = k$), make angles of 45° with the axes. Also where the curve cuts the asymptotes, $y = \frac{2}{3}k$. Hence the curve consists of three branches, as shown by the continuous lines through B, B_1 in Fig. 2, in which $OB_1 = -2k$.

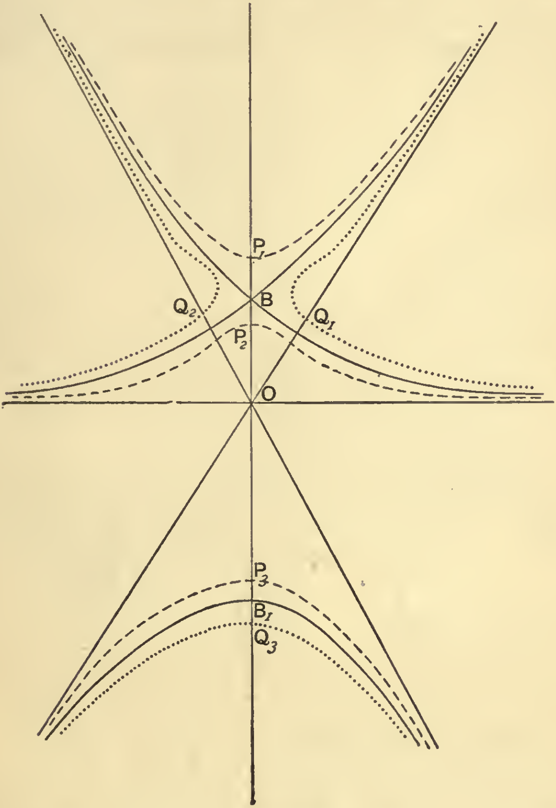


FIG. 2.

If $Y<2k^3$, that is, $<2\left(\frac{a}{3}\right)^{\frac{3}{2}}$, it will be easily seen that the curve has three branches, such as those shown by the broken lines (P_1, P_2, P_3) in Fig. 2: while if $Y>2k^3$ the three branches will be such as those shown by the dotted lines (Q_1, Q_2, Q_3) in the figure. The continuous transition from one form to the other as Y passes through the value $2k^3$ is also evident from the figure.

The curve $Y = -2k^3$ and the neighbouring curves, would be represented by the same diagram turned upside down, corresponding to the change of y into $-y$.

In the next diagram, Fig. 3, are shown the guiding curves through the double point B for both systems; also the hyperbola, vertices C, C', such that $OC = OC' = k\sqrt{3} = \sqrt{a}$, which, with the x axis, represents the curve $Y = 0$. The broken lines represent curves of the X system, the hyperbola, vertices B, B', such that $OB = OB' = k$, which, with the y axis, represents the curve $X = b$, and the three branches through A_1, A_2, A_3 corresponding to $X = 0$, b being supposed positive. The parts shaded by horizontal lines indicate the regions in which X is negative

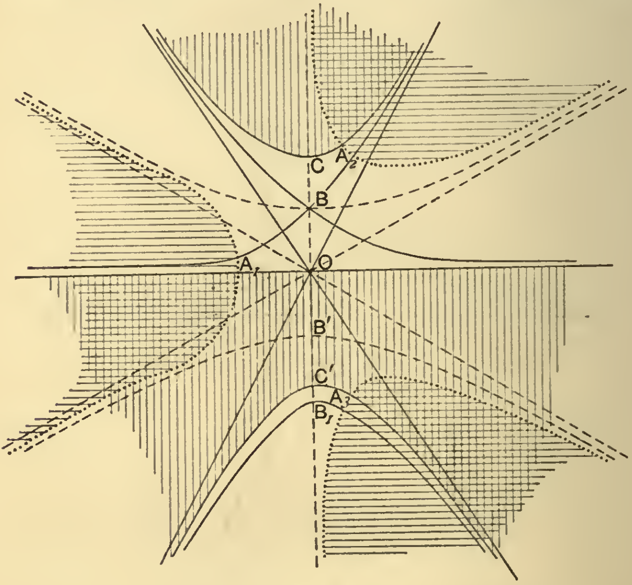


FIG. 3.

and those shaded by vertical lines the regions in which Y is negative. The points A_1, A_2, A_3, where the bounding curves of these regions intersect, are the three radical points, and hence it is obvious that in this case, where a and b are both positive, the equation $z^3 + az + b = 0$ has one negative and two complex roots.

The case, where b is negative, would be represented by turning the diagram through half a revolution about the y axis, corresponding to the change of x into $-x$.

In this case there is one positive and two complex roots. Hence, when a is positive, the roots of the equation $z^3+az+b=0$ are of the form -2α, $\alpha+i\beta$, $\alpha-i\beta$, where α is of the same sign as b.

II. Suppose $a = 0$. Then $fz = z^3 + b$. After the preceding discussion the diagram needs no explanation. It is useful as

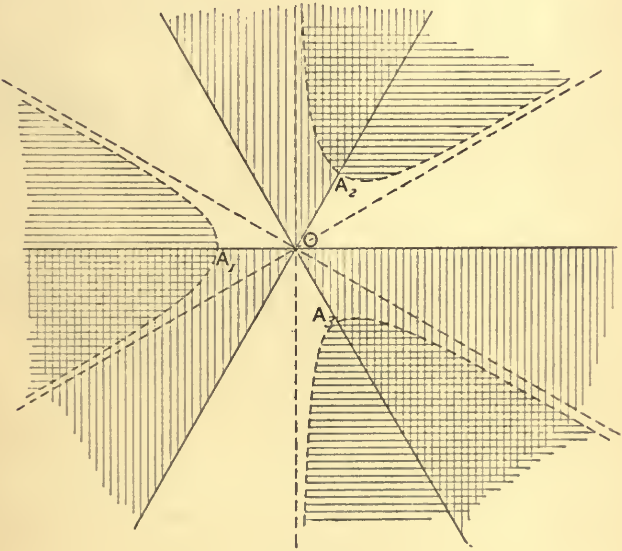


FIG. 4.

showing the continuous transition from the case of a positive to that of a negative. It is drawn for the case of b positive, and $\overline{OA_1}$, $\overline{OA_2}$, $\overline{OA_3}$ have equal tensors and are the three cube roots of $-b$. Observe the *triple* point in both systems at O, arising from the coincidence of B, B' with O, with which also C, C' and B_1 coincide.

III. Suppose a to be negative and put $-3k^2$ for a. Then the double points are $x = \pm k$, $y = 0$, and the curves passing through them are $Y = 0$, $X - b = \mp 2k^3$. The curve

$$X = b - 2k^3 \text{ or } x(x^2 - 3y^2 - 3k^2) = -2k^3$$

reduces to the form

$$(x + 2k)(x - k)^2 - 3xy^2 = 0.$$

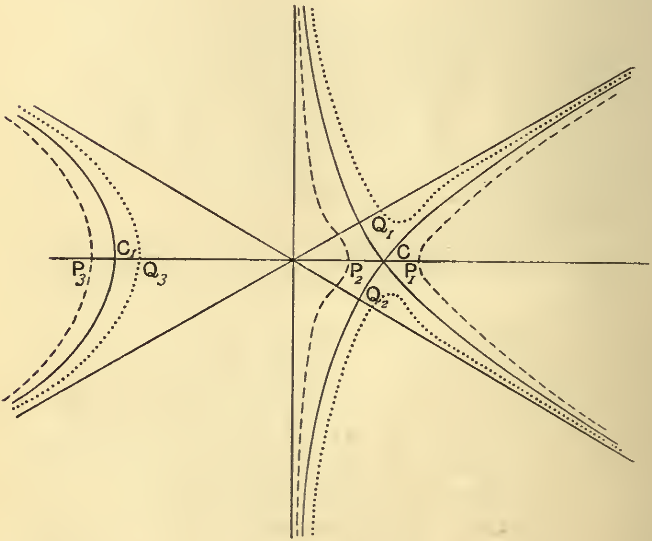


FIG. 5.

A similar discussion to that above for $Y=0$, when a is positive, leads for $X = b - 2k^3$ to the diagram (Fig. 5), which is in fact the same as Fig. 2 turned back through a right angle. The broken lines (P_1, P_2, P_3) correspond to $X > b - 2k^3$, and the dotted lines (Q_1, Q_2, Q_3) to $X < b - 2k^3$. If the figure is reversed by turning about the y axis, the curves correspond to the cases $X > = \text{ or } < b + 2k^3$.

The curve $Y=0$ is the hyperbola, vertices C, C' (Fig. 6) and the x axis, C, C' being thus double points. The broken lines (P_1, P_2, P_3) correspond to Y positive, and the dotted lines (Q_1, Q_2, Q_3) to Y negative.

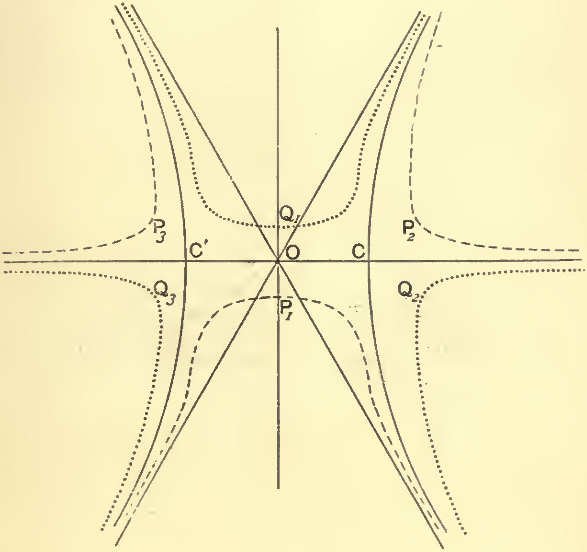


FIG. 6.

In the following diagram (Fig. 7), both the curves $X=0$, $Y=0$ are shown for the case, in which $b=2k^3$ or

$$\frac{b}{2} = \left(-\frac{a}{3}\right)^{\frac{3}{2}} \text{ or } \left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3 = 0.$$

In this case the double point at C is a double radical point, since there both $X=0$ and $Y=0$, the other radical point being C_1, such that $OC_1 = -2OC$, where the third branch of $X=0$ cuts the x axis. Hence the three roots of $fz=0$ are in this case scalar, two of them being equal. The different regions shaded horizontally and vertically show those for which X, Y respectively are negative. The hyperbola, vertices B, B', together with the y axis, represent the curve $X=b$.

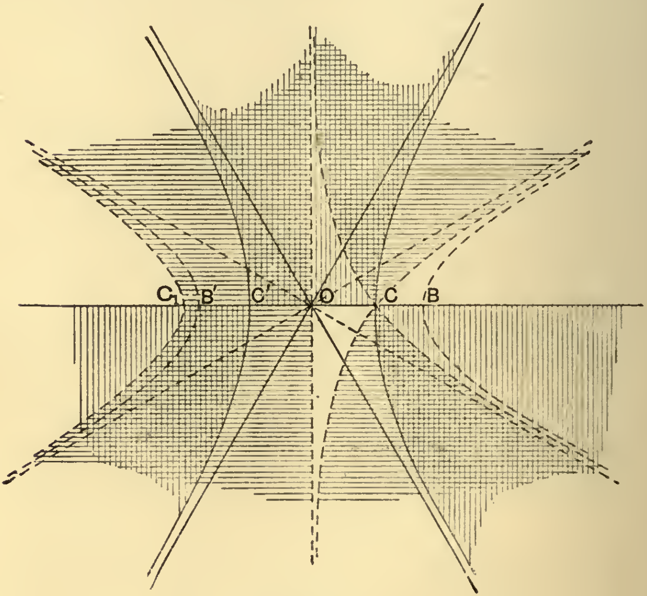


FIG. 7.

If $b > 2k^3$ or $< -2k^3$, so that $\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3$ is positive, the curve for $X=0$ passes into the negative region from C as shown in Fig. 5, and the two radical points which coincided with C (or with C') leave the x axis and lie on the hyperbola through C (or

C'), so that the two roots become complex of the form $\alpha + i\beta$, $\alpha - i\beta$.

If $b < 2k^3$, or $> -2k^3$, so that $\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3$ is negative, the curve for $X = 0$ passes into the positive regions of X and so intersects the x axis in three points as P_1, P_2, P_3 in Fig. 5. The three radical points are therefore on the x axis, and the three roots are scalar. Hence the well-known condition that the three roots of a cubic are all scalar, only if $\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3$ is negative.

9. *The Binomial Function* $z^n \pm 1$.

If $w = z^n \pm 1$, the relation between the variations of z and w may be very simply expressed as follows. (The proof, and diagrammatic illustrations for particular values of n, are left to the student.) To a circle described on the z plane about the origin as centre corresponds a circle on the w plane, the vector of whose centre is ± 1, in such a manner that to one complete circuit of the former correspond n complete circuits of the latter, any radius of the one inclined at the angle u to the prime vector corresponding to a radius of the other inclined at the angle nu. Also the ratio of the radii of any two such circles on the w plane is the n-plicate ratio of those of the corresponding circles on the z plane.

For other forms of w, to a circle similarly described on the z plane corresponds a curve on the w plane formed by the combination of two or more circular motions of greater or less complexity according to the number of the coefficients of the terms of fz.

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